B.Sc. II YEAR - ANALYTICAL GEOMETRY OF 3D AND VECTOR CALCULUS

Unit I : Rectangular Cartesian Coordinates in space Distance formula Direction ratio and cosines Angle between lines simple problems. Plane different forms of equation angle between two planes perpendicular distance from a point on a plane projection of a line or a point on a plane.

Unit II: Lines symmetrical form plane and a straight line The perpendicular from a point on a line Coplanar lines shortest distance between two skew lines and its equation. Sphere Different forms of equations- plane section the circle and its radius and centre tangent plane condition for tangency touching spheres common tangent plane point of orthogonality of intersection of two spheres.

Unit III: Vector differentiation Gradient, Divergence and Curl operators solenoidal and irrotational fields- formulas involving the Laplace operator.

Unit IV: Vector integration single scalar variables line, surface and volume integrals.

Unit V: Gausss Stokes and Greens theorems statements and verification only.

Books for Reference:

1. T.K. Manickavachagom Pillay and T. Natarajan, A text book of Analytical Geometry, Part II - Three Dimensions, S. Viswanathan (Printers and Publishers) Pvt. Ltd., 2010.

2. P. Duraipandian and Lakshmi Duraipandian, Analytical Geometry of 3D & Vector Calculus.

3. S. Arumugam and A. Thangapandi Isaac, Analytical Geometry 3D & Vector Calculus, New Gamma Publishing House, Palayamkottai, 2011.

4. K. Viswanathan, Vector Analysis.

Course material prepared by

Dr.S.Monikandan

Department of Mathematics

Manonmaniam Sundaranar University

Tirunelveli

Chapter 1

UNIT I

1.1 Rectangular cartesian coordinates in space

Let a rubber ball be dropped vertically in a room The point on the floor, where the ball strikes, can be uniquely determined with reference to axes, taken along the length and breadth of the room. However, when the ball bounces back vertically upward, the position of the ball in space at any moment cannot be determined with reference to two axes considered earlier. At any instant, the position of ball can be uniquely determined if in addition, we also know the height of the ball above the floor. If the height of the ball above the floor is 2.5 cm and the position of the point where it strikes the ground is given by $(5, 4)$, one way of describing the position of ball in space is with the help of these three numbers $(5, 4, 2.5)$. Thus, the position of a point (or an article) in space can be uniquely determined with the help of three numbers.

In this unit, we will discuss in details about the co-ordinate system and co-ordinates of a point in space, distance between two points in space, position of a point dividing the join of two points in a given ratio internally/externally and about the projection of a point/line in space.

Recall the example of a bouncing ball in a room where one corner of the room was considered as the origin.

It is not necessary to take a particular corner of the room as the origin. We could have taken any corner of the room (for the matter any point of the room) as origin of reference, and relative to that the coordinates of the point change. Thus, the origin can be taken arbitarily at any point of the room.

Let us start with an arbitrary point O in space and draw three mutually perpendicular lines $X'OX$, $Y'OY$ and $Z'OZ$ through O. The point O is called the origin of the co-ordinate system and the lines $X'OX$, $Y'OY$ and $Z'OZ$ are called the x-axis, the y-axis and the z-axis respectively. The positive direction of the axes are indicated by arrows on thick lines in Fig. 1.0. The plane determined by the X-axis and the Y-axis is called xy-plane (XOY plane) and similarly, yz-plane (YOZ-plane) and zx-plane (ZOX-plane) can be determined. These three planes are called co-ordinate planes. The three coordinate planes divide the whole space into eight parts called **octants**.

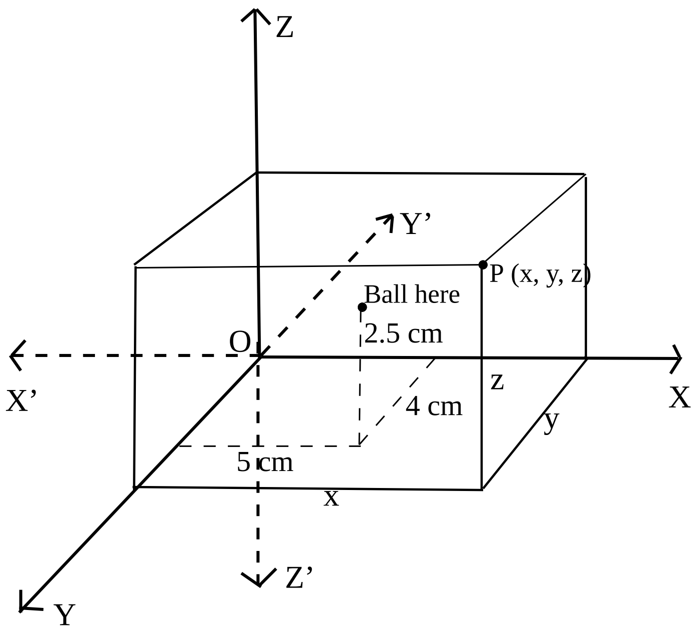


Fig. 1.0.

Now, let P be any point in space. Let x, y, z denote the perpendicular distances from P to the yz, zx and xy coordinate planes repectively. Then the three real numbers x, y, z are called the **rectangular cartesian coordinates** of P and the point P is represented by the ordered triple (x, y, z). Conversely, any ordered triple of real numbers (x, y, z) represents a unique point in space. Thus the set of points in space can be identified with the set $R^3 = \{(x, y, z) | x, y, z \in R\}$.

Note 1.1.1. *The space R^3 is divided into eight octants by the coordinate planes.*

Note 1.1.2. *The points on the x, y plane are of the form $(x, y, 0)$ the point on the yz plane are of the form (0,y,z) and the points on zx plane are of the form $(x, 0, z)$.*

Note 1.1.3. *The points on the x-axis are of the form $(x, 0, 0)$ and the points on the y-axis are of the form $(0, y, 0)$ and the points on the z-axis are of the form $(0, 0, z)$.*

1.2 Distance Formula

Theorem 1.2.1. *If $P(x_1, y_1, z_1)$ and Q(x_2, y_2, z_2) are two points then* $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

Proof. Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two given points.
Draw PM and QN perpendicular to the *xoy* plane. Then M is $(x_1, y_1, 0)$ and N is $(x_2, y_2, 0)$.

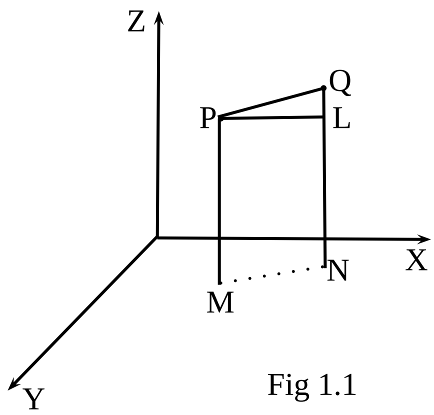


Fig 1.1

Therefore $MN^2 = (x_2-x_1)^2+(y_2-y_1)^2$ Draw PL perpendicular to QN. Since $PMNL$ is a rectangle $PL = MN$ and $MP = NL$. Now from right triangle PLQ we have

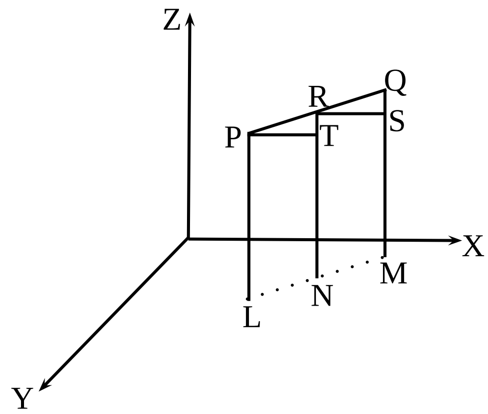
$$\begin{aligned} PQ^2 &= PL^2 + LQ^2 \\ &= MN^2 + (NQ - NL)^2 \\ &= MN^2 + (NQ - ML)^2 \\ &= [(x_2 - x_1)^2 + (y_2 - y_1)^2] + (z_2 - z_1)^2] \end{aligned}$$

$$\therefore \quad PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \qquad \square$$

Theorem 1.2.2. *The point of division R of the line joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ internally in the ratio $l : m$ is*

$$\left(\frac{lx_2+mx_1}{l+m}, \frac{ly_2+my_1}{l+m}, \frac{lz_2+mz_1}{l+m}\right)$$

Proof. Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be the two given points. Let R (x, y, z) be the point of division of PQ in the ratio $l : m$ internally.
Therefore $\dfrac{PR}{RQ} = \dfrac{L}{M}$.
Draw PL, RN and QM perpendicular to the xoy plane. Draw PT and RS perpendicular to NR and MQ respectively



Clearly, ΔPRT is similar to ΔRQS, we have

$$\frac{TR}{SQ} = \frac{PR}{RQ} = \frac{l}{m}$$

Therefore $\dfrac{z-z_1}{z_2-z} = \dfrac{l}{m}$

$$z = \frac{lz_2+mz_1}{l+m}$$

Similarly, we can prove that the other two coordinates of R are $x = \dfrac{lx_2+mx_1}{l+m}$ and $y = \dfrac{ly_2+my_1}{l+m}$ □

Corollary 1.2.3. *If R divides the line joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$* ***externally*** *in the ratio $l : m$ then R is* $\left(\dfrac{lx_2-mx_1}{l-m}, \dfrac{ly_2-my_1}{l-m}, \dfrac{lz_2-mz_1}{l-m}\right)$

Corollary 1.2.4. *The* ***midpoint*** *of the line joining* $P(x_1, y_1, z_1)$ *and* $Q(x_2, y_2, z_2)$ *is* $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$

Corollary 1.2.5. *The* ***centroid*** *of the triangle whose vertices are* (x_i, y_i, z_i) *i= 1,2,3 is* $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$.

1.3 DIRECTION COSINES AND DIRECTION RATIOS

Definition 1.3.1. *Let* α, β, γ *be the angles made by a straight line with the positive directions of the coordinate axes. These angles are called the* ***direction angles*** *and the cosines of these angles are called the* ***direction cosine*** (*d.c*) *of the line.*

Note 1.3.2. *The direction cosines of a line are usually denoted by* l, m, n *so that* $l = \cos\ \alpha$, $m = \cos\ \beta$, *and* $n = \cos\ \gamma$.

Note 1.3.3. *The direction cosines of the* x, y *and* z*-axis are respectively* $1, 0, 0$; $0, 1, 0$ *and* $0, 0, 1$.

Theorem 1.3.4. *If* l, m, n *are the d.c of a line the* $l^2 + m^2 + n^2 = 1$.

Proof. Consider the line λ which has the direction cosines l, m, n. Draw a line through O parallel to the line λ. Take any point $P(x, y, z)$ on the line λ. Let $OP = r$.
Then $r = \sqrt{x^2 + y^2 + z^2}$ $\cdots(1)$
Draw PN perpendicular to OX.
From right $\Delta\ ONP$, $\cos\ \alpha = \frac{x}{r}$. Similarly, $\cos\ \beta = \frac{y}{r}$ and $\cos\ \gamma = \frac{z}{r}$.

$$\begin{aligned}\therefore l^2 + m^2 + n^2 &= \cos^2\alpha + \cos^2\beta + \cos^2\gamma \\ &= \frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} \\ &= \frac{x^2 + y^2 + z^2}{r^2} \\ &= 1 \text{ (using (1))}\end{aligned}$$

Hence the theorem. □

Definition 1.3.5. *Any three numbers a, b, c which are proportional to d.c of a line are called the* ***direction ratios*** *(d.r) or* ***direction numbers*** *of the line. Hence $l = ak$; $m = bk$; $n = ck$ where k is a nonzero constant.*

1.3.1 The relation between direction cosines and direction ratios

If we know the direction ratios a, b, c of a line then we can find the direction cosines as follows. We have $l = ak$; $m = bk$; $n = ck$; for $k \neq 0$.

Now, $l^2 + m^2 + n^2 = 1$. Hence $k^2(a^2 + b^2 + c^2) = 1$

Therefore

$$k = \pm\frac{1}{\sqrt{a^2+b^2+c^2}}.$$

Therefore d.c are

$$\pm\frac{a}{\sqrt{a^2+b^2+c^2}}, \pm\frac{b}{\sqrt{a^2+b^2+c^2}}, \pm\frac{c}{\sqrt{a^2+b^2+c^2}},$$

where the positive or negative sign is taken throughout.

Theorem 1.3.6. *The direction ratios of the line joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $x_2 - x_1, y_2 - y_1, z_2 - z_1$.*

Proof. In Fig. 1.1, $LQ = z_2 - z_1$.

Let PQ make angles α, β, γ with the positive direction of the coordinate axes.

In the right-angled triangle $\Delta PLQ, \angle PQL = \gamma$.

$\therefore \cos\ \gamma = \frac{LQ}{PQ} = \frac{z_2 - z_1}{r}$ where $r = PQ$.

Similarly, $\cos\alpha = \frac{x_2 - x_1}{r}$ and $\cos\beta = \frac{y_2 - y_1}{r}$

$\therefore x_2 - x_1, y_2 - y_1, z_2 - z_1$ are the direction of PQ. □

Corollary 1.3.7. *If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are two points, then the d.c of the line PQ are*

$$\pm\frac{x_2 - x_1}{\sqrt{\sum(x_2 - x_1)^2}}, \pm\frac{y_2 - y_1}{\sqrt{\sum(x_2 - x_1)^2}}, \pm\frac{z_2 - z_1}{\sqrt{\sum(x_2 - x_1)^2}}$$

.

Theorem 1.3.8. *The angle between two lines whose direction cosines are l, m, n and l_1, m_1, n_1 respectively is given by* $\cos\ \theta = ll_1 + mm_1 + nn_1$.

Proof. Let OP and OQ be the two lines drawn through O and parallel to the given lines. Let θ be the angle between the lines.
Let $OP = r$ and $OQ = r_1$.
Therefore P is (lr, mr, nr) and Q is (l_1r_1, m_1r_1, n_1r_1).
In ΔOPQ, we have $PQ^2 = OP^2 + OQ^2 - 2\ OP\ OQ \cos\ \theta$ $\cdots (1)$
Therefore $PQ^2 = r^2 + r_1^2 - 2rr_1 \cos\theta$

$$\begin{aligned} \text{Also } PQ^2 &= (l_1r_1 - lr)^2 + (m_1r_1 - mr)^2 + (n_1r_1 - nr)^2 \\ &= r_1^2(l_1^2 + m_1^2 + n_1^2) + r^2(l^2 + m^2 + n^2) - 2rr_1(ll_1 + mm_1 + nn_1) \end{aligned}$$

$$= r^2 + r_1^2 - 2rr_1(ll_1 + mm_1 + nn_1) \qquad \cdots (2)$$

From(1) and (2), we get $\cos\theta = ll_1 + mm_1 + nn_1$. □

Corollary 1.3.9. $\sin\theta = \sqrt{(lm_1 - l_1m)^2 + (mn_1 - m_1n)^2 + (nl_1 - n_1l)^2}$

$$\begin{aligned} \sin^2\theta &= 1 - \cos^2\theta \\ &= 1 - (ll_1 + mm_1 + nn_1)^2 \\ &= (l + m^2 + n^2)(l_1^2 + m_1^2 + n_1^2) - (ll_1 + mm_1 + nn_1)^2 \\ &= (l^2m_1^2 - 2lml_1m_1 + l_1^2m^2) + (m^2n_1^2 - 2mnm_1n_1 + m_1^2n^2) \\ &\qquad + (n^2l_1^2 - 2nn_1ll_1 + n_1^2l^2) \\ \textit{Therefore} \quad \sin\theta &= \sqrt{(lm_1 - l_1m)^2 + (mn_1 - m_1n)^2 + (nl_1 - n_1l)^2} \end{aligned}$$

Corollary 1.3.10. *If a, b, c and a_1, b_1, c_1 are the direction ratios of the lines then the angle between the lines is given by* $\cos\ \theta = \frac{aa_1+bb_1+cc_1}{\sqrt{\sum a^2}\sqrt{\sum a_1^2}}$ *and*

$\sin\ \theta = \frac{\sqrt{(ab_1-a_1b)^2+(bc_1-b_1c)^2+(ca_1-c_1a)^2}}{\sqrt{a^2+b^2+c^2}\sqrt{a_1^2+b_1^2+c_1^2}}$

Corollary 1.3.11. *Two lines whose d.c are l, m, n and l_1, m_1, n_1 respectively are* ***perpendicular*** *if and only if $ll_1 + mm_1 + nn_1 = 0$.*

Corollary 1.3.12. *Two line whose d.r are a, b, c and a_1, b_1, c_1 are perpendicular if and only if $aa_1 + bb_1 + cc_1 = 0$*

Corollary 1.3.13. *Two lines with direction cosines l, m, n and l_1, m_1, n_1 are* ***parallel*** *if and only if $\frac{l}{l_1} = \frac{m}{m_1} = \frac{n}{n_1}$ and consequently if and only if $\frac{a}{a_1} = \frac{b}{b_1} = \frac{c}{c_1}$.*

Proof. The two lines are parallel $\Leftrightarrow \sin\theta = 0$

$$\begin{aligned}
&\Leftrightarrow \quad (lm_1 - l_1m)^2 + (mn_1 - m_1n)^2 + (nl_1 - n_1l)^2 = 0 \\
&\Leftrightarrow \quad lm_1 - l_1m = 0; mn_1 - m_1n = 0; nl_1 - n_1l = 0 \\
&\Leftrightarrow \quad \frac{l}{l_1} = \frac{m}{m_1} = \frac{n}{n_1}
\end{aligned}$$

consequently if and only if $\frac{a}{a_1} = \frac{b}{b_1} = \frac{c}{c_1}$. □

Area of the triangle with vertices $A(x_1, y_1, z_1);\ B(x_2, y_2, z_2);\ C(x_3, y_3, z_3)$

Let the area of the triangle ABC be Δ. Let the angles made by the plane of the ΔABC with the coordinate planes be α, β, γ respectively. Then l, m, n are the direction cosines of the normal to the plane containing triangle ABC so that $l^2 + m^2 + n^2 = 1$. Then, $\cos\alpha = l$; $\cos\beta = m$; $\cos\gamma = n$;

$\therefore \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \qquad \cdots (1)$

Let $A_1, B_1, C_1; A_2B_2C_2$ and $A_3B_3C_3$ be the orthogonal projections of the triangle ABC on the xy plane; zx plane; yz plane respectively. Then the vertices of $A_1B_1C_1$ are $(x_1, y_1, 0)$;$(x_2, y_2, 0)$;$(x_3, y_3, 0)$; the vertices of $A_2B_2C_2$ are $(x_1, 0, z_1)$;$(x_2, 0, z_2)$;$(x_3, 0, z_3)$ and the vertices of $A_3B_3C_3$ and $(0,y_1, z_1)$;$(0, y_2, z_2)$; $(0, y_3, z_3)$.

$$\therefore \Delta_1 = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}; \Delta_2 = \frac{1}{2}\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix} \text{ and } \Delta_3 = \frac{1}{2}\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}.$$

We know that projection of the area A enclosed by a curve in a plane is $A\ \cos\theta$

where θ is the angle between the plane of the curve containing the given area and the plane of projection. Since $\Delta_1, \Delta_2, \Delta_3$ are the orthogonal projections of Δ we have $\Delta_1 = \Delta\cos\alpha$; $\Delta_2 = \Delta\cos\theta$; $\Delta_3 = \Delta\cos\gamma$.

$$\begin{aligned}\Delta_1^2+\Delta_2^2+\Delta_3^2 &= \Delta^2(\cos^2\alpha+\cos^2\beta+\cos^2\gamma) = \Delta^2 \quad \text{(using (1))}\\ \therefore \Delta &= \sqrt{\Delta_1^2+\Delta_2^2+\Delta_3^2}\end{aligned}$$

1.4 Solved Problems.

Problem 1.4.1. Show that the point $(2,3,5)$, $(-1,5,-1)$ and $(4,-3,2)$ form an isosceles right-angled triangle.

Let the points be A, B, C respectively.

Then $AB^2 = (2+1)^2 + (3-5)^2 + (5+1)^2 = 49$.

$BC^2 = (-1-4)^2 + (5+3)^2 + (-1+2)^2 = 98$.

$CA^2 = (4-2)^2 + (-3-3)^2 + (2-5)^2 = 49$.

Therefore $AB = CA$ and $BC^2 = AB^2 + CA^2$.Hence $\angle A = 90°$.

Therefore ABC is an isosceles right-angled triangle.

Problem 1.4.2. The line joining $A\ (5,2,4)$ and $B(-4,3,5)$ meets the planes YOZ, XOY in C, D respectively. Find the coordinates of C and D and the ratios in which they divide AB.

The point which divides AB in the ratio $\lambda : 1$ has coordinates $\left(\frac{5-4\lambda}{1+\lambda}, \frac{2+3\lambda}{1+\lambda}, \frac{4+5\lambda}{1+\lambda}\right)$. If the point lies on the YOZ plane its x-coordinate must be zero and so $5-4\lambda = 0$, $\lambda = \frac{5}{4}$.

Therefore C is the point $\left(0, \frac{23}{9}, \frac{41}{9}\right)$.

Since λ is positive, C divides AB in ratio $5 : 4$.

If the point lies on the XOY plane, its z-coordinate must be zero and so $4 + 5\lambda = 0$, That is, $\lambda = -\frac{4}{5}$. D is therefore $(41, -2, 0)$ and it divides AB externally in the ratio $4 : 5$.

Problem 1.4.3. Find the direction cosines of the line joining the points $(3,-5,4)$ and $(1,-8,-2)$.

The direction cosines of the line are proportional to $3-1, -5+8, 4+2$.

That is , proportional to 2,3,6.

Let them be $2k, 3k, 6k$.

But $(2k)^2 + (3k)^2 + (6k)^2$=1.

That is, $49k^2$=1 i.e., $k = \pm\frac{1}{7}$.

Taking the positive value for k, the direction cosines of the line are $\frac{2}{7}, \frac{3}{7}, \frac{6}{7}$.

Problem 1.4.4. *Find the ratio in which the xy plane divides the line joining the points $A(7, 4, -2)$ and $B(8, -5, 3)$. Also find the point of division.*

Solution. Let AB meet the xy-plane at C. Let P be any point on AB dividing it in the ratio $k : 1$.

Then P is $\left(\frac{8k+7}{k+1}, \frac{-5k+4}{k+1}, \frac{3k-2}{k+1}\right)$ $\cdots(1)$

If this lies on the xy plane then z-coordinate of C must be zero.

$\therefore$ $\frac{3k-2}{k+1} = 0$. Hence $3k - 2 = 0$ so that $k = \frac{2}{3}$.

$\therefore C$ divides AB internally in the ratio 2:3.

$\therefore$ Substituting $k = \frac{2}{3}$ in (1), we get C is $\left(\frac{37}{5}, \frac{2}{5}, 0\right)$.

Problem 1.4.5. *Find the direction cosines of the line which is equally inclined to axes.*

Solution. Let the lines have direction cosines l, m, n where $l = \cos\alpha$; $m = \cos\beta$; $n = \cos\gamma$ where α, β, γ are the angles which the line makes with the positive direction of the x, y, z axes respectively.

Given that the line is equally inclined to the axes. Hence $\alpha = \beta = \gamma$.

We know, for the line, $l^2 + m^2 + n^2 = 1$.

$\Longrightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

$\Longrightarrow 3\cos^2\alpha = 1$. Hence $\cos^2\alpha = 1/3$, which implies $\cos\alpha = \pm(1/\sqrt{3})$.

$\therefore$ The direction cosines are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

Problem 1.4.6. *Find the d.c of the lines AB and CD where $A = (1, 2, -4)$, $B(2, 1, -3)$, $C(4, 6, -1)$ and $D(5, 7, 0)$. Hence find the acute angle between them.*

Solution. The $d.r$ of the line AB are $1 - 2, 2 - 1, -4 + 3$.

$\Longrightarrow d.r$ of AB are $-1, 1, -1$.

$\therefore$ *d.c* of AB are $\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$ (taking the positive value of the root sign).

The *d.r* of the line CD are $4-5, 6-7, -1-0$.

$\Longrightarrow$ *d.r* of CD are $-1, -1, -1$.

$\therefore$ *d.c* of CD are $\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$.

Let θ be the acute angle between AB and CD.

$\therefore \cos\theta = ll_1 + mm_1 + nn_1 = \frac{1}{3} - \frac{1}{3} + \frac{1}{3} = -\frac{1}{3}$

$\therefore \theta = \cos^{-}1(1/3)$ (since θ is acute)

Problem 1.4.7. *Show that the angle between two diagonals of a cube is* $\cos^{-}1(\frac{1}{\sqrt{3}})$.

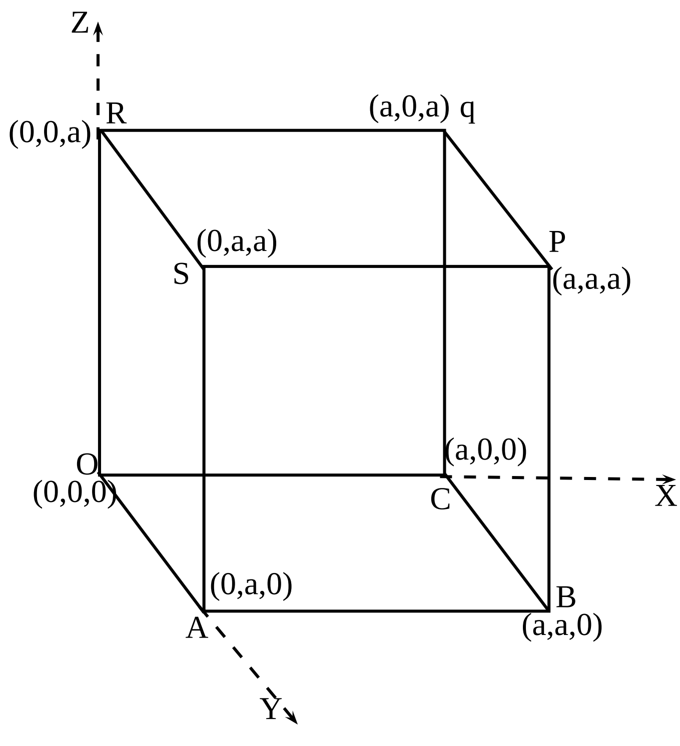
Solution. Let the length of each side of the cube be a.

OP and RB is a pair of diagonals where $O = (0,0,0)$, $P = (a,a,a)$, $R = (0,0,a)$ and $B = (a,a,0)$.

$\therefore$ *d.r* of OP are a, a, a.

Hence *d.c* of OP are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

d.r of RB are $a, a, -a$.



Hence the *d.c* of RB are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$

Therefore the angle θ between the two diagonals is given by $\cos\,\theta = \frac{1}{\sqrt{3}}\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}\left(-\frac{1}{\sqrt{3}}\right)$.

$=\frac{1}{3}+\frac{1}{3}-\frac{1}{3}=\frac{1}{3}$

$\therefore \theta = \cos^{-1}(\frac{1}{3})$.

Problem 1.4.8. *A line makes angles* $\alpha, \beta, \gamma, \delta$ *with four diagonals of a cube. Prove that* $sin^2\alpha + sin^2\beta + sin^2\gamma + sin^2\delta = \frac{8}{3}$.

Solution. Refer the above figure.

The four diagnols of the cube are *OP, RB, AQ* and *SC*.

The direction cosines of

$$\begin{aligned} OP &= \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \\ RB &= \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \\ AQ &= \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \\ SC &= \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \end{aligned}$$

Let the *d.c* of the given line be *l,m,n*.

Let it make angles $\alpha, \beta, \gamma, \delta$ with these four diagonals respectively.

$$\begin{aligned} \therefore \cos\alpha &= \frac{l}{\sqrt{3}} + \frac{m}{\sqrt{3}} + \frac{n}{\sqrt{3}} \\ \cos\beta &= \frac{l}{\sqrt{3}} + \frac{m}{\sqrt{3}} - \frac{n}{\sqrt{3}} \\ \cos\gamma &= \frac{l}{\sqrt{3}} - \frac{m}{\sqrt{3}} + \frac{n}{\sqrt{3}} \\ \cos\delta &= \frac{l}{\sqrt{3}} - \frac{m}{\sqrt{3}} - \frac{n}{\sqrt{3}} \end{aligned}$$

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}(l^2+m^2+n^2) = \frac{4}{3}$

$\therefore (1-\sin^2\alpha) + (1-\sin^2\beta) + (1-\sin^2\gamma) + (1-\sin^2\delta) = \frac{4}{3}$

$\therefore \sin^2\alpha + \sin^2\beta + \sin^2\gamma + \sin^2\delta = 4 - \frac{4}{3} = \frac{8}{3}$

Problem 1.4.9. *A line makes* 30° *and* 120° *with the positive directions of the* x *and* y *axes respectively. What angle does it make with the positive direction of the z-axis?*

Solution. $l = \cos\ 30° = \frac{\sqrt{3}}{2}$ and $m = \cos\ 120 = -\frac{1}{2}$.

Now $l^2 + m^2 + n^2 = 1$.

$\therefore \left(\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + n^2 = 1$

$\therefore n^2 = 1 - \frac{3}{4} - \frac{1}{4} = 0$.

Therefore $n = 0$. Hence $\cos\gamma$=0.

$\therefore \gamma = 90°$.

Therefore the line makes $90°$ with the positive direction of the z-axis.

Problem 1.4.10. *Find the locus of P such that $PA^2 + PB^2 = k^2$ where A is $(3, 4, 5)$ and B is $(-2, 3, -7)$ and k is constant.*

Solution. Let P be (x_0, y_0, z_0) be a point on the locus. $PA^2 + PB^2 = k^2$

$\Rightarrow (x_0 - 3)^2 + (y_0 - 4)^2 + (z_0 - 5)^2 + (x_0 + 1)^2 + (y_0 - 3)^2 + (Z_0 - 7)^2 = k^2$.

$\Rightarrow 2x_0^2 + 2y_0^2 + 2z_0^2 - 4x_0 - 14y_0 + 4z_0 + 109 - k^2 = 0$.

Therefore the locus of (x_0, y_0, z_0) is $2x^2 + 2y^2 + 2z^2 - 4x - 14y + 4z + 109 - k^2 = 0$.

Problem 1.4.11. *Show that (i) the lines joining the midpoints of the opposite edges of a tetrahedron are concurrent;*

(ii) their point of concurrency is the centroid of the tetrahedron.

Solution. Let $ABCD$ be the tetrahedron whose vertices are $A(x_1, y_1, z_1)$; $B(x_2, y_2, z_2)$; $C(x_3, y_3, z_3)$ and $D(x_4, y_4, z_4)$.

(i) $AC, BD; AB, CD; AD, BC$ are the three pairs of opposite edges.

Let M_1, M_2, M_3,

M_4, M_5, M_6 be their midpoints respectively.

M_1 is $\left(\frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}, \frac{z_1+z_3}{2}\right)$; M_2 is $\left(\frac{x_2+x_4}{2}, \frac{y_2+y_4}{2}, \frac{z_2+z_4}{2}\right)$.

Therefore the mid points of the line M_1M_2 is $\left(\frac{\sum x_1}{4}, \frac{\sum y_1}{4}, \frac{\sum z_1}{4}\right)$.

The symmetry of this result shows that midpoints of M_3M_4 and M_5M_6 is the same as the midpoint of M_1M_2.

Hence M_1M_2, M_3M_4, M_5M_6 are concurrent.

(ii) We know that the centroid G of the tetrahedron divides the line joining each vertex to the centroid of the opposite triangular face in the ratio 3:1

Let H be the centroid of the triangular face BCD.

Therefore H is $\left(\frac{x_2+x_3+x_4}{3}, \frac{y_2+y_3+y_4}{3}, \frac{z_2+2_3+2_4}{3}\right)$
We have $AG : GH = 3 : 1$.
Therefore the x-coordinate of G is $\frac{3\left(\frac{x_2+x_3+x_4}{3}\right)+1.x_1}{3+1} = \frac{\sum x_1}{4}$
Therefore G is $\left(\frac{\sum x_1}{4}, \frac{\sum y_1}{4}, \frac{\sum z_1}{4}\right)$.
Hence the result follows.

Problem 1.4.12. *If two pairs of opposite edges of a tetrahedron are perpendicular. Show that third pair is also perpendicular.*

Solution. Let $AB, CD; AC, BD; AD, BC$ be the three pairs of opposite edges of a tetrahedron $ABCD$; let the first two pairs be perpendicular.
That is $AB \perp CD$ and $AC \perp BD$.
We claim that $AD \perp BC$.
Let (x_i, y_i, z_i) i=1,2,3,4 be the vertices of the tetrahedron ABCD.
The d.r of AB are $x_2 - x_1, y_2 - y_1, z_2 - z_1$ and the d.r of CD are $x_4 - x_3, y_4 - y_3, z_4 - z_3$
$AB \perp CD \Rightarrow (x_2 - x_1)(x_4 - x_3) + (y_2 - y_1) + (y_4 - y_3) + (z_2 - z_1)(z_4 - z_3) = 0$ ...(1)
$AC \perp BD \Rightarrow (x_3 - x_1)(x_4 - x_2) + (y_3 - y_1) + (y_4 - y_2) + (z_3 - z_1)(z_4 - z_2) = 0$ ...(2)
Now, $(x_2 - x_1)(x_4 - x_3) + (y_2 - y_1) + (y_4 - y_3) + (z_2 - z_1)(z_4 - z_3)$

$$\begin{aligned}
&= x_2x_4 - x_2x_3 - x_1x_4 + x_1x_3 - x_3x_4 + x_3x_2 + x_1x_4 - x_1x_2 \\
&= x_2x_4 + x_1x_3 - x_3x_4 - x_1x_2 \\
&= x_4(x_2 - x_3) - x_1(x_2 - x_3) \\
&= (x_4 - x_1)(x_2 - x_3) \qquad \cdots (3)
\end{aligned}$$

We get similar results by interchanging y and z with x in (3).
Subtracting (2) from (1) and using (3), we get
$(x_4 - x_1)(x_2 - x_3) + (y_4 - y_1)(y_2 - y_3) + (z_4 - z_1)(z_2 - z_3) = 0$
$\therefore AD \perp BC$. Hence the result.

Problem 1.4.13. *If the sum of the squares of opposite sides of a tetrahedron are equal prove that its opposite sides are at right angles.*

Solution. Let $O(0, 0, 0)$; $A(x_1, y_1, z_1)$; $B(x_2, y_2, z_2)$; $C(x_3, y_3, z_3)$ be the coordinates of the vertices of the tetrahedron.

Then $OA, BC; OB, AC; OC, AB$ are the three pairs of opposite sides of the tetrahedron.

Given $OA^2 + BC^2 = OB^2 + CA^2 = OC^2 + AB^2$.

We have to prove OA, OB, OC are perpendicular to BC, AC, AB respectively.

Taking $OA^2 + BC^2 = OB^2 + CA^2$, we get

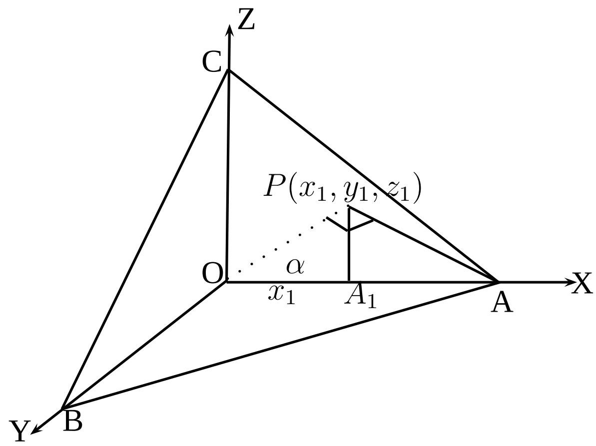
$x_1^2 + y_1^2 + z_1^2 + (x_3 - x_2)^2 + (y_3 - y_2)^2 + (z_3 - z_2)^2 = x_2^2 + y_2^2 + z_2^2 + (x_3 - x_2)^2 + (y_3 - y_2)^2 + (z_3 - z_2)^2$.

$\therefore 2(x_2x_3 + y_2y_3 + z_2z_3) = 2(x_1x_3 + y_1y_3 + z_1z_3)$.

That is, $x_3(x_2 - x_1) + y_3(y_2 - y_1) + z_3(z_2 - z_1) = 0$

Therefore OC is perpendicular to AB. Similarly, we can prove OB is perpendicular to AC and OC is perpendicular to AB.

Problem 1.4.14. *From a point $P(x_1, y_1, z_1)$ a plane is drawn at right angles to OP meeting the coordinate axes at A,B,C. Prove that the area of the triangle ABC is $\frac{r^5}{2x_1y_1z_1}$ where r is the algebraic distance of OP.*



Solution. From the right triangle OPA, we have $OA = r\sec\alpha$ where α is the angle which line makes with the positive direction of the x-axis.

Therefore A is $(r\sec\alpha, 0, 0)$. similarly, $B(0, r\sec\beta, o)$; C$(0, 0, \sec\gamma)$

From the right-angled triangle OA_1P, we have $x_1 = r\cos\alpha$.

$\therefore$ $\sec\alpha = r/x_1$. Similarly $\sec\beta = r/y_1$ and $\sec\gamma = r/x_1$.

Therefore, the vertices of the triangles ABC are $A(r^2/x_1, 0, 0); B(0, r^2/y_1, 0); C(0, 0, r^2/z_1)$

Now, arc of $\Delta ABC = \Delta = \sqrt{\Delta_1^2 + \Delta_2^2 + \Delta_3^2}$ where

$$\Delta_1 = \frac{1}{2}\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ r^2/y_1 & 0 & 1 \\ 0 & r^2/z_1 & 1 \end{vmatrix} = \frac{r^4}{2y_1z_1}$$

Similarly $\Delta_2 = \frac{r^4}{2z_1x_1}$ and $\Delta_3 = \frac{r^4}{2x_1y_1}$

$\therefore \Delta = \left(\frac{1}{2}\right)\sqrt{\left(\frac{r^8}{x_1^2y_1^2} + \frac{r^8}{z_1^2y_1^2} + \frac{r^8}{x_1^2z_1^2}\right)} = \left(\frac{r^4}{2}\right)\sqrt{\left(\frac{x_1^2+y_1^2+z_1^2}{x_1^2y_1^2z_1^2}\right)}$

$= \frac{r^5}{2x_1y_1z_1}$ (since $x_1^2 + y_1^2 + z_1^2 = r^2$)

Problem 1.4.15. *Show that the straight lines whose d.c are given by $2l - m + 2n$=0 and $lm + mn + nl = 0$ are at right angles.*

Solution. Given $2l - m + 2n = 0$

$\therefore 2(\frac{l}{n}) - (\frac{m}{n}) + 2 = 0$ $\cdots(1)$

Also given $lm + mn + nl = 0$.

$\therefore (\frac{l}{n})(\frac{m}{n}) + (\frac{m}{n} + (\frac{l}{n}) = 0$ $\cdots(2)$

From (1), we get $\frac{m}{n} = 2(\frac{l}{n}) + 2$ $\cdots(3)$

Substituting(3) in (2) we get $2(\frac{l}{n})^2 + 2(\frac{l}{n}) + 2(\frac{l}{n}) + 2 + (\frac{l}{n}) = 0$

$2(\frac{l}{n})^2 + 5(\frac{l}{n}) + 2 = 0$

This is a quadratic equation in $\frac{l}{n}$ and solving we get $\frac{l}{n} = -2, -\frac{1}{2}$.

From(3), we get $\frac{m}{n} = -2, 1$

If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of the two given lines then we have

$\frac{l_1}{n_1} = -2; \frac{m_1}{n_1} = -2.$

$\frac{l_2}{n_2} = -\frac{1}{2}; \frac{m_2}{n_2} = 1$

$\therefore \frac{l_1l_2}{n_1n_2} = 1$ and $\frac{m_1m_2}{n_1n_2} = -2$

$\therefore l_1l_2 = n_1n_2$ and $m_1m_2 = -2n_1n_2$

Now, $l_1l_2 + m_1m_2 + n_1n_2 = n_1n_2 - 2n_1n_2 + n_1n_2 = 0$

Therefore the two lines are perpendicular.

Exercises 1.4.16.

1. Find the distance between the following pairs of points $(1, -3, 2)$ and $(2, 5, -4)$.

2. Find the point dividing line joining $(3, 2, 1)$ and $(3, -3, 6)$ in the ratio $3 : 2$ 'internally and externally.

3. Find the direction cosines of the lines whose direction ratios are (i) $3, -4, 5$ (ii) $2, -1, 3$

4. Find the direction ratios and direction cosines of the line joining the points $(1, 2, -1)$ to $(2, 1, 3)$.

5. Find the direction cosines of the lines which makes $45°$ with OX, $60°$ with OY and $120°$ with OZ.

1.5 plane

In this section, we study several forms of the equation of a plane in R^3.

Definition 1.5.1. *A plane in R^3 is defined to be the locus of a point (x, y, z) satisfying a linear equation of the form $ax + by + cz = 0$ where a, b, c are not all zero.*

Theorem 1.5.2. *Equation of a plane pasing through a given point (x_1, y_1, z_1) and having a normal whose d.r are a, b, c is given by $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$.*

Proof. Let $A(x_1, y_1, z_1)$ be a given point on the plane. Let LM be a normal to the plane. The $d.r$ of LM are a, b, c. Let $P(x, y, z)$ be any point on the plane. Then AP is perpendicular to LM.
Also $d.r$ of AP are $x - x_1, y - y_1, z - z_1$.

$$\therefore \quad a(x - x_1) + b(y - y_1) + c(z - z_1) = 0 \qquad \cdots (1)$$

Since $P(x, y, z)$ is arbitrary, equation (1) represents the equation of the plane. □

Theorem 1.5.3. *The equation of the plane passing through the points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ is given by*

$$\begin{pmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{pmatrix} = 0. \qquad \cdots (1)$$

Proof. Let the equation of the plane be $ax + by + cz + d = 0$. $\cdots(2)$

Since A, B, C lie on this plane we have

$ax_1 + by_1 + cz_1 + d = 0$ $\cdots(3)$

$ax_2 + by_2 + cz_2 + d = 0$ $\cdots(4)$

$ax_3 + by_3 + cz_3 + d = 0$ $\cdots(5)$

Eliminating the constants a, b, c from (2)(3)(4) and (5) we get the result(1). □

Note 1.5.4. *In numerical problems, it is convenient to solve the three equation (3), (4),and (5) in terms of d directly and get the equation of the plane on substitution in (2).*

Aliter. The equation of any plane passing through (x_1, y_1, z_1) can be written in the form

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ $\cdots(1)$

If this passes through (x_2, y_2, z_2) and (x_3, y_3, z_3) also, we have

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ $\cdots(2)$

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ $\cdots(3)$

Eliminating a, b, c from (1), (2) and (3), we have

$$\begin{pmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{pmatrix} = 0$$

which is the equation of the plane passing through (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3).

Note 1.5.5. *To verify whether four points are coplanar we have to find the equation of the planes passing through any three points and check whether the fourth point lie on it or not. Equivalently, the four points are coplanar if*

$$\begin{pmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{pmatrix} = 0$$

Theorem 1.5.6. ***(Intercepts form)*** *The equation of the plane having intercepts a, b, c with the coordinate axes is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.*

Proof. Let the equation plane be $Ax + By + Cz + D = 0$ $\cdots(1)$

Let this plane meet the coordinate axes OX, OY, OZ at P, Q, R respectively.

$\therefore\ OP = a;\ OQ = b;\ OR = c.$

Therefore P is $(a, 0, 0)$; Q is $(0, b, 0)$ and R is $(0, 0, c)$.

Since P, Q, R lie on the plane we have

$Aa + D = 0; Bb + d = 0; Cc + D = 0.$

$\therefore\ A = -\frac{D}{a}; B = -\frac{D}{b}; C = -\frac{D}{c}.$

Therefore, equation (1) becomes $-\frac{D}{a}x - \frac{D}{b}y - -\frac{D}{c}z + D = 0.$

That is, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$

Hence the theorem. □

Theorem 1.5.7. ***(Normal form)*** *The equation of a plane can be written as $lx + my + nz = p$ where l, m, n are the d.c of the normal to the plane and p is the length of the perpendicular from the origin to plane.*

Proof. Let the plane meet the coordinate axes at A, B, C with intercepts a, b, c;

Therefore, the equation of the plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c}$=1 $\cdots(1)$

Let the length of the perpendiuclar OD from O to the plane be p.

Now, $l = cos\angle DOA = \frac{OD}{OA} = \frac{p}{a}$.

Therefore $a = \frac{p}{l}$. Similarly $b = \frac{p}{m}$ and $c = \frac{p}{n}$.

Therefore equation (1) of the plane becomes $\frac{lx}{p} + \frac{my}{p} + \frac{nz}{p} = 1.$

That is, $lx + my + nz = p.$ □

Note 1.5.8. *The above equation of the plane can also be written as $x\cos\ \alpha + y\cos\ \beta + z\cos\ \gamma = p$ where α, β, γ are the angles which the normal to the plane makes with the coordinate axes.*

1.5.1 Transformation to the normal form

The general equation of the plane $ax + by + cz + d = 0$. $\cdots(1)$

where $a^2 + b^2 + c^2 \neq 0$ can be transformed to the normal form

$lx + my + nz = p$. $\cdots(2)$

Equations (1) and (2) represent the same plane if $\frac{a}{l} = \frac{b}{m} = \frac{c}{n} = \frac{-d}{p} = k$ (say)

Therefore, $l = \frac{a}{k}, m = \frac{b}{k}, n = \frac{c}{k}$ and p=-$\frac{d}{k}$.

Since $l^2 + m^2 + n^2 = 1$ we get $k = \pm\sqrt{a}^2 + b^2 + c^2$

$\therefore l = \pm\frac{a}{\sqrt{a^2+b^2+c^2}}$; $m = \pm\frac{b}{\sqrt{a^2+b^2+c^2}}$;

$n = \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$; and $p = \pm\frac{-d}{\sqrt{a^2+b^2+c^2}}$ $\cdots(3)$

Now we choose the sign of k opposite to that of d so as to make p positive.

Substituting (3) in (2), we obtain the required normal form.

Note 1.5.9. *For the plane $ax + by + cz + d = 0$ a, b, c are d.r of the normal to plane and $\pm\frac{a}{\sqrt{(\sum a^2)}}, \pm\frac{b}{\sqrt{(\sum b^2)}}, \pm\frac{c}{\sqrt{(\sum a^2)}}$ (with suitable sign so that p is always positive) denote the d.c of the normal to the plane.*

Example 1.5.10. The $d.r$ of the normal to plane $2x - 3y + 6z + 7 = 0$ are $2, -3, 6$. Hence the direction cosines are $-\frac{2}{7}, \frac{3}{7}, -\frac{6}{7}$ and the length of the normal from the origin to the plane is $\frac{7}{7} = 1$

1.6 ANGLE BETWEEN TWO PLANES

Definition 1.6.1. ***Angle between two planes*** *is defined to the angle between the normals to them from any point.*

Theorem 1.6.2. *The angle θ between planes $ax + by + cz + d = 0$ and $a_1x + b_1y + c_1z + d_1 = 0$ is given by $cos\theta = \pm\left[\frac{aa_1+bb_1+cc_1}{\sqrt{(\sum a^2)}\sqrt{(\sum a_1^2)}}\right]$*

Proof. The $d.c$ of the normal to plane $ax + by + cz + d = 0$ are

$\frac{a}{\sqrt{(\sum a^2)}}, \frac{b}{\sqrt{(\sum a^2)}}, \frac{c}{\sqrt{(\sum a^2)}}$.

The $d.c$ of the normal to the plane $a_1x + b_1y + c_1z + d_1 = 0$ are

$\frac{a_1}{\sqrt{(\sum a_1^2)}}, \frac{b_1}{\sqrt{(\sum a_1^2)}}, \frac{c_1}{\sqrt{(\sum a_1^2)}}$

Therefore the angle between the planes is given by $\cos\theta = \pm\left[\frac{aa_1+bb_1+cc_1}{\sqrt{(\sum a^2)}\sqrt{(\sum a_1^2)}}\right]$ □

Corollary 1.6.3. *The planes $ax + by + cz + d = 0$; $a_1x + b_1y + c_1z + d_1 = 0$ are at right angles if and only if $aa_1 + bb_1 + cc_1$=0*

Corollary 1.6.4. *The planes $ax + by + cz + d = 0$ and $a_1x + b_1y + c_1z + d_1 = 0$ are parallel if and only if $\frac{a}{a_1} = \frac{b}{b_1} = \frac{c}{c_1}$ Hence the equation of a plane parallel to $ax + by + cz + d = 0$ is of the form $ax + by + cz + k = 0$.*

Theorem 1.6.5. *Length of the perpendicular from a point $A(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is $\pm\left[\frac{ax_1+by_1+cz_1+d}{\sqrt{(a^2+b^2+c^2)}}\right]$.*

Proof. Let $lx + my + nz+ = p$ be the normal form of the plane $ax + by + cz + d = 0$. Therefore $l = \pm\frac{a}{\sqrt{(\sum a^2)}}$, $m = \pm\frac{b}{\sqrt{(\sum a^2)}}$,n=$\pm\frac{c}{\sqrt{(\sum a^2)}}$, and $p = \pm\frac{d}{\sqrt{(\sum a^2)}}$.
Now equation of the plane through the given point $A(x_1, y_1, z_1)$ and parallel to the plane $lx + my + nz = p$ is given by $lx + my + nz = p_1$
where p_1 is the length of the perpendicular from the origin to the plane.
Since (x_1, y_1, z_1) lies on (1) we have $lx_1 + my_1 + nz_1 = p_1$
Now the length of the perpendicular from (x_1, y_1, z_1) to the given plane is

$$\begin{aligned} p - p_1 &= p - lx_1 - my_1 - nz_1 \\ &= \pm\frac{d}{\sqrt{(\sum a^2)}} - \left\{\pm\frac{ax_1}{\sqrt{(\sum a^2)}}, \pm\frac{by_1}{\sqrt{(\sum a^2)}}, \pm\frac{cz_1}{\sqrt{(\sum a^2)}}\right\}. \\ &= \pm\left[\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{(\sum a^2)}}\right] \end{aligned}$$

Hence the result. □

Theorem 1.6.6. *Equation of a plane through the line of intersection of two given planes $\pi_1 = 0$ and $\pi_2 = 0$ is $\pi_1 + \lambda\pi_2 = 0$ (λ is a constant).*

Proof. Let $\pi_1 = a_1x + b_1y + c_1z + d_1 = 0$ $\cdots(1)$

$\pi_2 = a_2x + b_2y + c_2z + d_2 = 0$ $\cdots(2)$

be the two equations of the two planes.

Consider the equation $\pi_1 + \lambda\pi_2 = 0$

That is, $(a_1x + b_1y + c_1z + d_1)(a_2x + b_2y + c_2z + d_2) = 0$ $\cdots(3)$

Equation (3) is of the first degree in x, y, z and hence represents a plane.

further any point (x_1, y_1, z_1) satisfying (1) and (2) also satisfies (3). Hence (3) passes through the line of intersection (1) and (2).

Two sides of a plane. Consider a plane and two points A and B not lying in the plane. Then the points A, B may lie on (i) opposite sides of the plane or (ii) in the same side of the plane.

If A, B lie on either side of the plane, the segment AB has common point with the plane whereas if A, B lie in the same side of the plane the segment AB does not have a common point with the plane.

We proceed to find a criterion for two given points to lie on the same or different sides of a given plane. □

Theorem 1.6.7. *Two points $A(x_1, y_1, z_1)$,$B(x_2, y_2, z_2)$ lie on the same or different sides of the plane $ax + by + cz + d = 0$ according as $ax_1 + by_1 + cz_1 + d$ and $ax_2 + by_2 + cz_2 + d$ are of the same or different signs.*

Proof. Let the line AB meet the given plane at P. Let P divide AB in the ratio $k : 1$. If k is positive P divides AB internally and if k is negative P divides AB externally.

That is., if k is positive A, B lie on opposite sides of the plane and if k is negative A, B lie on the same side of the plane.

P is $\left(\frac{kx_2+x_1}{k+1}, \frac{ky_2+y_1}{k+1}, \frac{kz_2+z_1}{k+1}\right)$ and P lies on the plane.

Hence we have $a\left(\frac{kx_2+x_1}{k+1}\right) + b\left(\frac{ky_2+y_1}{k+1}\right) + c\left(\frac{kz_2+z_1}{k+1}\right) + d = 0$

$\therefore k(ax_2 + by_2 + cz_2 + d) + (ax_1 + by_1 + cz_1 + d) = 0$

$\therefore k = -\left(\frac{ax_1+by_1+cz_1+d}{ax_2+by_2+cz_2+d}\right)$

k is negative if $ax_1 + by_1 + cz_1 + d$ and $ax_2 + by_2 + cz_2 + d$ are of opposite signs.

Hence the result follows. □

Example 1.6.8. The origin and $(2, -3, 7)$ lie on the same side of the plane $2x - 3y + 2z + 8 = 0$.

For, by substituting the two points in the expression $2x - 3y + 2z + 8$ the values are of same signs.

Example 1.6.9. $(2, 1, 1)$ and $(2, 5, -1)$ lie on different sides of the plane $x - 2y + 3z + 4 = 0$.(verify)

Example 1.6.10. Find the equation to the plane through $(3, 4, 5)$ parallel to the plane $2x + 3yz = 0$.

The equation to any plane parallel to this plane is

$2x + 3y - +k = 0$.

If it passes through the point $(3, 4, 5)$.

$2(3) + 3(4) - 5 + k = 0$

That is,$k = -13$.

Hence the equation of the required plane is $2x + 3y - z - 13 = 0$.

Example 1.6.11. Find the angle between the planes $2x - y + z = 6, x + y + 2z = 3$.

The direction cosines of the normals to the planes are proportional to 2,-1,1 and 1,1,2 respectively.

If θ be the angle between the planes then,

$\cos\theta = \frac{2-1+2}{\sqrt{(2^2+(-1)^2+1^2)}\sqrt{(1^2+1^2+2^2)}} = \frac{3}{\sqrt{6}\sqrt{6}} = \frac{1}{2}$

$\therefore \theta = \frac{\pi}{3}$

Example 1.6.12. Find the equation of the plane which passes through the point $(-1, 3, 2)$ and perpendicular to the two planes $x + 2y + 2z = 5$, $3x + 3y + 2z = 8$.

Let the equation of the required plane be $Ax + By + Cz + d = 0$.

It passes through the point $(-1, 3, 2)$.

$$\therefore \quad -A + 3B + 2C + D = 0 \qquad \cdots (1)$$

The plane is perpendicular to the planes $x + 2y + 2z = 5$ and $3x + 3y + 2z = 8$

$\therefore A + 2B + 2C = 0$ $\cdots(2)$

$3A + 3B + 2C = 0$ $\cdots(3)$

From the equations(2)(3) we get $\frac{A}{-2} = \frac{B}{4} = \frac{C}{-3}$

Let each be equal to k.

Then $A = -2k$, $B = 4k$, $C = -3k$.

Substituting the value of A, B, C in equation(1), we get $D = -8k$.

Hence the eqution of the plane is $-2kx + 4ky - 3kz - 8k = 0$.

That is, $2x - 4y + 3z + 8 = 0$.

Example 1.6.13. Find the distance between the parallel planes $2x - 2y - z + 3 = 0$ and $4x - 4y + 2z + 5 = 0$.

Find a point on the plane $2x - 2y - z + 3 = 0$ and the distance

between the two parallel planes is the perpendicular distance from

that point to the plane $4x - 4y + 2z + 5 = 0$.

The first plane meets the z-axis at the point $(0, 0, -3)$.

The length of the perpendicular from $(0, 0, -3)$ to the plane

$4x - 4y + 2z + 5 = 0$ is $\pm\frac{-6+5}{\sqrt{(4^2+4^2+2^2)}} = \pm\frac{1}{6}$.

Hence the distance between the parallel planes is $\frac{1}{6}$.

1.7 Projection of a line

Definition 1.7.1. *(i) The projection of a point on a line is the foot of the perpendiculars drawn from the point on the line.*

(ii) The projection of a finite straight line on another is the portion of the second line intercepted between the projections of the extremities of the finite line on the second.

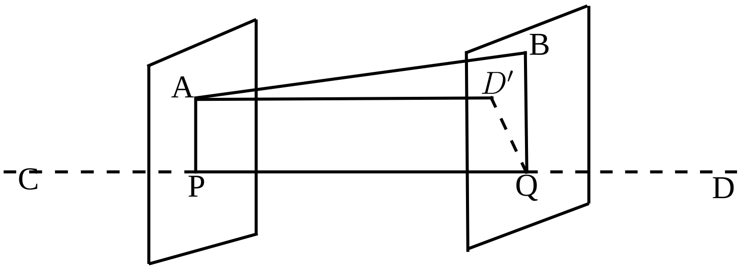
Thus, the projection on AB on a line l is A_1B_1, where A_1 and B_1 are the feet of the perpendiculars drawn from the points A, B on l.

Result 1.7.2. *The projection of a finite straight line AB on another straight line CD is $AB\cos\theta$ where θ is the angle between AB and CD.*

Proof. Draw AD' parallel to CD. Then the angle between AB and AD', That is, $\angle BAD'$ is θ. Through A and B draw two planes, each perpendicular to CD, the first one cutting CD and AD' at P and A and the second cutting them at Q and D' respectively.

AD' is parallel to PQ; AP is parallel to $D'Q$.

$$\therefore \quad AD' = PQ.$$



PQ is the projection of the line AB and CD.

But BD' is perpendicular to AD.

$\therefore \quad AB\cos\theta = AD'$.

$$\begin{aligned}\therefore \text{ Projection of } AB \text{ on } CD &= BQ\\ &= AD'\\ &= AB\cos\theta\end{aligned}$$

□

1.7.1 Solved problems

Problem 1.7.3. *Find the equation of the plane passing through* $(1,1,0)$, $(1,2,1)$ *and* $(-2,2,-1)$

Solution. Let the equation of the required plane be $ax + by + cz + d = 0$ $\cdots(1)$

Since the given points lie on it we have

$$a + b + d = 0 \quad \cdots (2)$$

$$a + 2b + c + d = 0 \quad \cdots (3)$$

$$-2a + 2b - c + d = 0 \quad \cdots (4)$$

$$(2) - (3) \Rightarrow -b - c = 0 \quad \cdots (5)$$

$$(3) - (4) \Rightarrow 3a + 2c = 0 \quad \cdots (6)$$

From (5) and (6) we have $\frac{a}{-2} = \frac{b}{-3} = \frac{c}{3} = k(\text{say})$

Therefore $a = -2k; b = -3k; c = 3k$.

Substituting in (2) we have $d = 5k$.

Therefore (1) becomes $-2x - 3y + 3z + 5 = 0$.

Problem 1.7.4. *Find the equation of the plane passing through* $(2, 2, 1)$ *and* $(9, 3, 6)$ *and perpendicular to the plane* $2x + 6y + 6z = 9$.

Solution. Equation of the plane passing through $(2, 2, 1)$ is

$a(x - 2)b(y - 2) + c(z - 1) = 0 \quad \cdots (1)$

where a, b, c are the $d.r$ of the normal to the plane to determined.

Since $(9, 3, 6)$ lies on this plane, we have $7a + b + 5c = 0 \quad \cdots (2)$

Since the plane (1) is perpendicular to $2x + 6y + 6z = 9$,

we have $2a + 6b + 6c = 0 \quad \cdots (3)$

Solving (2) and (3) we have $\frac{a}{-24} = \frac{b}{-32} = \frac{c}{40}$

Therefore $\frac{a}{3} = \frac{b}{4} = \frac{c}{-5} = k$ (say)

Therefore $a = 3k, b = 4k, c = -5k$.

Substituting in (1), we get $3(x - 2) + 4(y - 2) - 5(z - 1) = 0$

Therefore $3x + 4y - 5z - 9 = 0$.

Problem 1.7.5. *Find the equation of the plane throught* $(2, 3, -4)$ *and* $(1, -1, 3)$ *and parallel to the x-axis.*

Solution. Equation of the plane passing through $(2, 3, -4)$ is

$a(x - 2) + b(y - 3) + c(z + 4) = 0 \quad \cdots (1)$

where a, b, c are the d.r of the normal to the plane which is to be determined.

Since $(1, -1, 3)$ also lies on the we have

$-a - 4b + 7c = 0$ $\cdots(2)$

Since the plane (1) is parallel to the x-axis its normal is perpendicular to the x-axis whose $d.r$ are 1,0,0.

$\therefore \quad a.1 + b.0 + c.0 = 0 \Rightarrow a = 0$ $\cdots(3)$

From (2) and (3), we get $a = 0, b = 7k$, and $c = 4k$.

Substituting in (1) we get the equation of the required place as

$7k(y - 3) + 4k(z + 4) = 0.$

$\therefore \quad 7y + 4z - 5 = 0$ is the equation of the required plane.

Problem 1.7.6. *Find the equation of the plane which passes through the point $(3, -2, 4)$ and is perpendicular to the line joining the points $(2, 3, 5)$ and $(1, -2, 3)$.*

Solution. Since the plane is perpendicular to the line joining $A(2, 3, 5)$ and $B(1, -2, 3)$, the line AB is normal to the plane.

The $d.r$ of the normal AB are 1,5,2.

Therefore the equation of the required plane is $1(x - 2) + 5(y + 2) + 2(z - 4) = 0$.

That is, $x + 5y + 2z - 1 = 0$.

Problem 1.7.7. *Find the equation of the plane which passes through the point $(1, -2, 1)$ and is perpendicular to each of the planes $3x + y + z - 2 = 0$ and $x - 2y + z + 4 = 0$.*

Solution. Let the equation of the plane be $ax + by + cz + d = 0$ $\cdots(1)$

It passes through $(1, -2, 1)$. Hence we get

$a - 2b + c + d = 0$ $\cdots(2)$

Since (1) is perpendicular to the planes $3x + y + z - 2 = 0$ and $x - 2y + z + 4 = 0$,

we have $3a + b + c = 0$ $\cdots(3)$

$a - 2b + c = 0$ $\cdots(4)$

Therefore $\frac{a}{3} = \frac{b}{-2} = \frac{c}{-7}$=($k$ say)

$\therefore a = 3k; b = 2k; c = -7k.$

Substituting in (2), we get $d = 0$.

Therefore the equation of the required plane is $3x - 2y - 7z = 0$.

Problem 1.7.8. *The foot of the perpendicular from the origin to a plane is* $(2, -1, 2)$. *Find the equation of the plane.*

Solution. Let $ax + by + cz + d = 0$ be the equation of the required plane. We know a, b, c are d.r of the normal to the plane. Since $P(2, -1, 2)$ is the foot of the perpendicular from the origin O to the plane, OP is the normal to that plane. Hence d.r of normal to the plane are $2, -1, 2$.

Therefore the equation of the plane becomes $2x - y + 2z + d = 0$.

Since $(2, -1, 2)$ lies on it we have $4 + 1 + 4 + d = 0$. Hence $d = -9$.

$\therefore 2x - y + 2z - 9 = 0$ is the equation of the required plane.

Problem 1.7.9. *Find the coordinate of the foot of the perpendicular drwan from the origin to the plane* $2x - 3y + z - 7 = 0$.

Solution. Let $P(x_1, y_1, z_1)$ be the foot of the perpendicular from the origin.

Since $P(x_1, y_1, z_1)$ lies on the plane we have $2x_1 - 3y_1 + z_1 - 7 = 0 \qquad \cdots (1)$

The direction ratios of OP are $x_1 - 0, y_1 - 0, z_1 - 0$.

That is, d.r of OP are x_1, y_1, z_1.

OP is normal to the given plane whose direction ratios are $2, -3, 1$.

Therefore $\frac{x_1}{2} = \frac{y_1}{-3} = \frac{z_1}{1} = k$ (say)

Therefore $x_1 = 2k$; $y_1 = -3k$; $z_1 = k$

Substituting in (1), we get $4k + 9k + k = 7$. Hence $k = \frac{1}{2}$

$\therefore x_1 = 1, y_1 = -\frac{3}{2}, z_1 = \frac{1}{2}$.

Therefore the foot of the perpendicular is $\left(1, -\frac{3}{2}, \frac{1}{2}\right)$.

Problem 1.7.10. *A plane meets the coordinate axes at A, B, C such that the centroid of the ΔABC is the point (α, β, γ). Show that the equation of the plane is* $\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 3$.

Solution. Let the equation of the plane be $ax + by + cz + d = 0 \qquad \cdots (1)$

Since it meets the x-axis at A we get $A = \left(-\frac{d}{a}, 0, 0\right)$.

Similarly, $B = \left(0, -\frac{d}{b}, 0\right)$ and $C = \left(0, 0, -\frac{d}{c}\right)$

Centroid of ΔABC is $\left(-\frac{d}{3a}, -\frac{d}{3b}, -\frac{d}{3c}\right)$

But the centroid of ΔABC is given to be (α, β, γ).

$\therefore \frac{-d}{3a} = \alpha; -\frac{d}{3b} = \beta;$ and $-\frac{d}{3c} = \gamma$.

$\therefore a = -\frac{d}{3\alpha}; b = -\frac{d}{3\beta}$ and $c = -\frac{d}{3\gamma}$.

Therefore the equation of the required plne (1) becomes

$-\frac{d}{3\alpha}x - \frac{d}{3\beta}y - \frac{d}{3\gamma}z + d = 0$. That is, $\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 3$

Problem 1.7.11. *A moving plane passes through a fixed point (α, β, γ) and intersects the coordinate axes at A, B, C. Show that the locus of the centroid of the ΔABC is $\frac{\alpha}{x} + \frac{\beta}{y} + \frac{\gamma}{z} = 3$.*

Solution. Let A be $(a, 0, 0), B(0, 0, b)$ and $C(0, 0, c)$.

Let (x_1, y_1, z_1) be the centroid of the triangle ABC.

$\therefore x_1 = \frac{a}{3}; y_1 = \frac{b}{3}; z_1 = \frac{c}{3}$ $\cdots (1)$

Now the equation of the plane ABC is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

It passes through the fixed point (α, β, γ).

$\therefore \frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c} = 1$. Hence $\frac{\alpha}{3x_1} + \frac{\beta}{3y_1} + \frac{\gamma}{3z_1} = 1$ by (1).

That is $\frac{\alpha}{x_1} + \frac{\beta}{y_1} + \frac{\gamma}{z_1} = 3$

Therefore the locus of (x_1, y_1, z_1) is $\frac{\alpha}{x} + \frac{\beta}{y} + \frac{\gamma}{z} = 3$.

Problem 1.7.12. *Find the equation of the plane through the intersection of the planes $3x - y + 2z - 4 = 0$, $x + y + z - 2 = 0$ and passing through the point $(2, 2, 1)$.*

Solution. The required plane is $3x - y + 2z - 4 + a(x + y + z - 2) = 0$ where a is to be determined.

Since $(2, 2, 1)$ lies on it, we have $6 - 2 + 2 - 4 + a(2 + 2 + 1 - 2) = 0$

$\therefore 2 + 3a = 0$. Hence $a = -\frac{2}{3}$.

Therefore the equation of the required plane is

$3x - y + 2z - 4 - \frac{2}{3}(x + y + z - 2) = 0$

That is, $7x - 5y + 4z - 8 = 0$.

Problem 1.7.13. *Find the equation of the required plane through the intersection of the planes $x + 3y - z = 4$ and $2x + 2y + 2z = 1$ which is perpendicular to the plane $x + y - 4z = 0$.*

Solution. The equation of the required plane is of the form

$x + 3y - z - 4 + a(2x + 2y + 2z - 1) = 1$

$\therefore (1 + 2a)x + (3 + 2a)y + (-1 + 2a)z - 4 - a = 0$

Since it is perpendicular to $x + y - 4z = 0$, we have

$\therefore -4a + 8 = 0$ Hence $a = 2$.

Therefore the equation of the required plane is

$x + 3y - z - 4 + 2(2x + 2y + 2z - 1) = 0.$

That is, $5x + 7y + 3z - 6 = 0$.

Problem 1.7.14. *Find the equation of the plane which is the rotation by an angle α of $lx + my = 0$ about its line of intersection with $z = 0$.*

Solution. The required plane is the plane passing through the intersection of the two planes $lx + my = 0$ and $z = 0$ and hence its equation is $lx + my + \lambda z = 0$ for some λ to be determined.

Given α is the angle between the planes $lx + my = 0$ and $lx + my + \lambda z = 0$.

$$\begin{aligned}\therefore \quad \cos\alpha &= \frac{l^2 + m^2 + \lambda(0)}{\sqrt{l^2 + m^2}\sqrt{l^2 + m^2 + \lambda^2}}\\ &\cos^2\alpha(l^2 + m^2)(l^2 + m^2 + \lambda^2) = (l^2 + m^2)^2\\ &[\cos^2\alpha(l^2 + m^2 + \lambda^2)] = (l^2 + m^2)\\ &\lambda^2\cos^2\alpha = (l^2 + m^2)(1 - \cos^2\alpha)\\ &\lambda^2 = (l^2 + m^2)\tan^2\alpha.\end{aligned}$$

Hence $\lambda = \pm\sqrt{l^2 + m^2} \;\; tan\alpha$

Therefore the equation of the required planes are $lx + my \pm [\sqrt{l^2 + m^2} \;\; \tan\alpha] = 0$.

Exercises 1.7.15.

1. Find the angle between the planes $x - y + 2z - 9 = 0$ and $2x + y + z = 7$.

2. Find the equation of the plane which passes through the point (2,-4,5) and is parallel to the plane $4x + 2y - 7z + 6 = 0$.

3. Find the equation of the plane passing through the points (1,2,3) and (-4,1,-2) and perpendicular to the plane $7x + 2y - z + 3 = 0$.

Chapter 2

UNIT II

2.1 Lines

We obtain different forms of equation of a straight line in space.

1.Non-symmetric form. We know that two planes in general intersect in a line. Hence a line in space can be represented by two linear equations.

$\pi_1 : a_1x + b_1y + c_1z + d_1 = 0$ and $\pi_2 : a_2x + b_2y + c_2z + d_2 = 0$.

2.Symmetric form. We can write the equations of a line if we know its direction cosines and a point on it.

Let $A(x_1, y_1, z_1)$ be a given point on the line. Let l, m, n be the $d.c$ of the line. Let $P(x, y, z)$ be any point on the line.

Therefore the direction ratios of AP are $x - x_1, y - y_1, z - z_1$.

Since the $d.c$ are l, m, n we have $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} = r$ $\cdots(1)$

Hence $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} = r$ represents the equations of the given line.

Note 2.1.1. Any point on the line (1) is $(x_1 + lr, y_1 + mr, z_1 + nr)$.

Note 2.1.2. The equation of a line passing through (x_1, y_1, z_1) and having direction ratios (a, b, c) are also given by $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} = r$

Note 2.1.3. The equations of a line given in non symmetric form can be converted to symmetric form as follows.

Let the equation of the two planes be $\pi_1 : a_1x + b_1y + c_1z + d_1 = 0$ and $\pi_2 : a_2x + b_2y + c_2z + d_2 = 0$. Let $\pi_1 = 0$ and $\pi_2 = 0$ intersect along a line $L : \pi_1 = \pi_2 = 0$. Let $d.r$ of L be l, m, n.

Since the line L lies on both $\pi_1 = 0$ and $\pi_2 = 0$ the normals to the planes'are perpendicular. Hence $a_1l + b_1m + c_1n = 0; a_2l + b_2m + c_2n = 0$

$$\therefore \frac{l}{b_1c_2 - b_2c_1} = \frac{m}{c_1a_2 - c_2a_1} = \frac{n}{a_1b_2 - a_2b_1}$$

Therefore the $d.r$ of L are $b_1c_2 - b_2c_1, c_1a_2 - c_2a_1, a_1b_2 - a_2b_1$.

We now find a point A on L by considering the point where it meets the plane $z = 0$ (xy-plane), (say). It is got from equations $a_1x + b_1y + d_1 = 0$ and $a_2x + b_2y + d_2 = 0$.

$$\therefore \frac{x}{b_1d_2 - b_2d_1} = \frac{y}{d_1a_2 - d_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

$$\therefore \quad A \text{ is } \left(\frac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1}, \frac{d_1a_2 - d_2a_1}{a_1b_2 - a_2b_1}, 0\right)$$

Hence the equation of the line L in symmetric form is

$$\frac{x - \left(\frac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1}\right)}{b_1c_2 - b_2c_1} = \frac{y - \left(\frac{d_1a_2 - d_2a_1}{a_1b_2 - a_2b_1}\right)}{c_1a_2 - c_2a_1} = \frac{z - 0}{a_1b_2 - a_2b_1}$$

Instead of finding the point where the given line meet the plane $z = 0$, we can also find the point where L meets plane $x = 0$ or $y = 0$.

3.Two-points form. Equation of straight line can be obtained when two points on the line are known.

If A(x_1, y_1, z_1) and B(x_2, y_2, z_2) are two points on a line, then the direction ratios of the line are $x_2 - x_1, y_2 - y_2, z_2 - z_1$.

Therefore the equation of the line is $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$.

Problem 2.1.4. *Find the direction cosines of the line* $\frac{2x+1}{3} = \frac{4y-3}{1} = \frac{2z-3}{0}$. *Also find a point on it.*

Solution. The given line can be rewritten as

$\frac{2\left(x+\frac{1}{2}\right)}{3} = \frac{4\left(y-\frac{3}{4}\right)}{1} = \frac{2\left(z-\frac{3}{2}\right)}{0}$.

That is, $\frac{x+\frac{1}{2}}{3/2} = \frac{y-\frac{3}{4}}{1/4} = \frac{z-\frac{3}{2}}{0}$. Hence the direction ratios are $\left(\frac{3}{2}, \frac{1}{4}, 0\right)$.

Therefore the direction cosines are $\left(\frac{6}{\sqrt{37}}, \frac{1}{\sqrt{37}}, 0\right)$.

A point on the line is $\left(-\frac{1}{2}, \frac{3}{4}, \frac{3}{2}\right)$.

Problem 2.1.5. *Find the value of k so that the lines* $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ *and* $\frac{x-1}{3k} = \frac{y-5}{1} = \frac{z-6}{-5}$ *may be perpendicular to each other.*

Solution. The direction ratios of the lines are $-3, 2k, 2$ and $3k, 1, -5$.

Since the lines are perpendicular, we have $(-3)3k + (2k)1 + 2(-5) = 0$

Hence $k = -10/7$.

Problem 2.1.6. *Prove that the lines* $x + y - z = 5; 9x - 5y + z = 4$ *and* $6x - 8y + 4z = 3; x + 8y - 6z + 7 = 0$ *are parallel.*

Solution. Let a, b, c be the direction ratios of the line determined by the planes $x + y - z = 5$; $9x - 5y + z = 4$.

Since the line is perpendicular to the normals of the above two planes, we have

$a + b - c = 0$

$9a - 5b + c = 0$.

$\therefore \frac{a}{-4} = \frac{b}{-10} = \frac{c}{-14}$

That is,$\frac{a}{2} = \frac{b}{5} = \frac{c}{7}$.

Therefore the direction ratios of the first lines are 2,5,7.

Similarly, we can prove that the direction ratios of the second lines are 2,5,7.

Hence the direction ratios of the two lines are proportional. Hence they are parallel.

Problem 2.1.7. *Find the angle between the lines*

$x - 2y + z = 0 = x + y - z - 3$ *and* $x + 2y + z - 5 = 0 = 8x + 12y + 5z$.

Solution. The first line is the intersection of the two planes $x - 2y + z = 0$ and $x + y - z - 3 = 0$. Let a, b, c be direction ratios of the line. Since the line is perpendicular to the normals of the above two planes

We have $a - 2b + = 0$ $\cdots(1)$

$a + b - c = 0$ $\cdots(2)$

$\therefore \frac{a}{1} = \frac{b}{2} = \frac{c}{3}$.

Therefore the direction ratios of the first line are $1, 2, 3$.

Let the direction ratios of the second line be a_1, b_1, c_1.

$\therefore a_1 + 2b_1 + c_1 = 0$

$8a_1 + 12b_1 + 5c_1 = 0$. Hence $\frac{a_1}{-2} = \frac{b_1}{3} = \frac{c_1}{-4}$.

Therefore the $d.r$ of the second lines are $-2, 3, -4$.

Therefore the angle θ between the two lines is given by

$$\begin{aligned}\cos\theta &= \frac{aa_1 + bb_1 + cc_1}{\sqrt{(\sum a^2)}\sqrt{(\sum a_1^2)}} \\ &= \frac{-2+6-12}{\sqrt{14}\sqrt{29}} = \frac{-8}{\sqrt{14}\sqrt{29}}\end{aligned}$$

$$\therefore \theta = \cos^{-1}\left(\frac{8}{\sqrt{406}}\right). \qquad (acute \quad angle)$$

Problem 2.1.8. *Find in symmetrical form the equations of the line given by* $x + 5y - z = 7; 2x - 5y + 3z + 1 = 0$.

Solution. The required line is the intersection of the planes

$x + 5y - z - 7 = 0 \qquad \cdots (1)$

$2x - 5y + 3z + 1 = 0 \qquad \cdots (2)$

Let a, b, c be the direction ratios of the line. Since the line is perpendicular to the normal of (1) and (2), we have

$a + 5b - c = 0 \qquad \cdots (3)$

$2a - 5b + 3c = 0 \qquad \cdots (4)$

$\therefore \frac{a}{10} = \frac{b}{-5} = \frac{c}{-15}$. Hence $d.r$ are $2, -1, -3$.

We now find one point on the line.

The line meets the xy-plane $z = 0$ at the point $(x, y, 0)$ where (x, y) satisfy the equations $x + 5y = 7 \qquad \cdots (5)$

$2x - 5y = -1 \qquad \cdots (6)$

Solving (5) and (6), we get $x = 2, y = 1$.

Therefore a point on the line is $(2, 1, 0)$.

Therefore the equations of the lines are $\frac{x-2}{2} = \frac{y-1}{-1} = \frac{z}{-3}$.

Problem 2.1.9. *Find the coordinate of the point of intersection of the lines* $\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z+3}{4}$ *with the plane* $2x + 4y - z + 1 = 0$.

Solution. Any point on the given line is $P(1 + 2r, 2 - 3r, -3 + 4r)$.

If P lies on the plane $2x + 4y - z + 1 = 0$, then

$2(1 + 2r) + 4(2 - 3r) - (-3 + 4r) + 1 = 0$.

Therefore $4r - 12r - 4r + 2 + 8 + 3 + 1 = 0$

Therefore $-12r = -14$. Hence $r = 7/6$.

Therefore P is $\left(1 + \frac{7}{3}, 2 - \frac{7}{2}, -3 + \frac{14}{3}\right)$

Therefore P is $\left(\frac{10}{3}, \frac{-3}{2}, \frac{5}{3}\right)$

Problem 2.1.10. *Find the perpendicular distance of the point* $P(1, 1, 1)$ *from the line* $\frac{x-2}{3} = \frac{y+3}{2} = \frac{z}{-1}$. *Also find the foot of the perpendicular.*

Solution. Let A be the foot of the perpendicular form $P(1, 1, 1)$ on the line.

Therefore AP is perpendicular to the line.

The direction ratios of the lines are $3, 2, -1$.

The coordinates of A can be taken as $(2 + 3r, -3 + 2r, -r)$

Therefore the direction ratios of AP are $1 + 3r, -4 + 2r, -r - 1$.

Since AP is perpendicular to the line we have

$(1 + 3r)3 + (-4 + 2r)2 + (-r - 1)(-1) = 0$.

Therefore $14r = 4$. Hence $r = \frac{2}{7}$

Therefore A is $\left(2 + \frac{6}{7}, -3 + \frac{4}{7}, -\frac{2}{7}\right) = \left(\frac{20}{7}, \frac{-17}{7}, -\frac{2}{7}\right)$.

Therefore the foot of the perpendicular is $\left(\frac{20}{7}, \frac{-17}{7}, -\frac{2}{7}\right)$.

$$\begin{aligned}
\therefore\ AP^2 &= \left(\frac{20}{7} - 1\right)^2 + \left(\frac{-17}{7} - 1\right)^2 + \left(-\frac{2}{7} - 1\right). \\
&= \left(\frac{13}{7}\right)^2 + \left(-\frac{24}{7}\right)^2 + \left(-\frac{9}{7}\right)^2 = \frac{118}{7} \\
\therefore\ AP &= \sqrt{\left(\frac{118}{7}\right)}
\end{aligned}$$

Problem 2.1.11. *Find the point where the line* $\frac{x-2}{2} = \frac{y-4}{-3} = \frac{z+6}{4}$ *meets the plane* $2x + 4y - z - 2 = 0$.

Solution. Let $\frac{x-2}{2} = \frac{y-4}{-3} = \frac{z+6}{4} = r$.

Therefore the coordinates of any point on the line are

$(2+2r, 4-3r, -6+4r)$.

If this point lies on the plane $2x+4y-z-2=0$, we get

$2(2+2r)+4(4-3r)-(-6+4r)-2=0$.

That is, $r=2$

Hence the coordinates of the required point are $(6,-2,2)$.

Problem 2.1.12. *Find the foot of the perpendicular from the origin on the line* $3x-y-z-4=0=4x-3y-2z+2$.

Solution. Let L be the line of intersection of the given planes. Let a, b, c be the d.r of the line L. Since L is perpendicular to the normal of both the planes we have

$3a-b-c=0$; $4a-3b-2c=0$.

$\therefore \frac{a}{-1} = \frac{b}{2} = \frac{c}{-5}$.

Hence d.r of the L are -1,2,-5.

Let A be the point of intersection of L with xy-plane, $z=0$.

The coordinates of A are obtained by solving $3x-y=4$ and $4x-3y=-22$.

Therefore A is $(14/5, 22/5, 0)$.

Hence the equation of the line L is $\frac{x-(14/5)}{-1} = \frac{y-(22/5)}{2} = \frac{z}{-5}$

Any point P on the line L is $P(-r+14/5, 2r+22/5, -5r)$

The $d.r$ of OP are $-r+14/5, 2r+22/5, -5r$.

Suppose P is the foot of the perpendicular from O to the line L.

Then, OP is perpendicular is to L gives

$-1(-r+14/5)+2(2r+22/5)-5(-5r)=0$.

$\therefore 30r=-6$. Hence $r=-1/5$

Therefore P is $(3,4,1)$

Problem 2.1.13. *Find the image of the point* $(2,3,4)$ *under the reflection in the plane* $x-2y+5z=6$.

Solution. Let $P(2,3,4)$ be the given point and let P' be its image under the reflection in the plane $x-2y+5z=6$.

The $d.r$ of the normal to the plane are 1, -2, 5.

Therefore the $d.r$ of PP' are also 1, -2, 5.

Hence the equations of the line are $\frac{x-2}{1} = \frac{y-3}{-2} = \frac{z-4}{5}$

Therefore P' is of the form $(2+r, 3-2r, 4+5r)$.

Mid point of PP' is $Q(2+r/2, 3-r, 4+5r/2)$.

Since Q lies on the plane (1) we have $(2+r/2) - 2(3-r) + 5(4+5r/2) = 6$

$\therefore 2 + r/2 - 6 + 2r + 20 + 25r/2 = 6$

Therefore $15r = -10$. Hence $r = -2/3$.

Therefore P' is $(2-2/3, 3+4/3, 4-10/3)$.

That is, P' is $\left(\frac{4}{3}, \frac{13}{3}, \frac{2}{3}\right)$.

Problem 2.1.14. *Find the image of the point* $(1,3,4)$ *under the reflection under the plane 2x-y+z+3=0. Hence prove that the image of the line* $\frac{x-1}{1} = \frac{y-3}{-2} = \frac{z-4}{-3}$ *is* $\frac{x+3}{1} = \frac{y-5}{-5} = \frac{z-2}{-10}$.

Solution. Let $P(1,3,4)$ be the given point and P' be its image in the plane

$2x - y + z + 3 = 0 \quad \cdots (1)$

The direction ratios of the normal to the plane 2,-1,1.

Therefore the direction ratios of PP' are also 2,-1,1.

Hence the equations of the line PP' are $\frac{x-1}{2} = \frac{y-3}{-1} = \frac{z-4}{1}$

The coordinates of P' are $(1+2r, 3-r, 4+r)$ for some r.

Mid point of PP' is Q$\left(\frac{1+2r+1}{2}, \frac{3-4+3}{3}, \frac{4+r+4}{2}\right)$

That is, Q is $(r+1, 3-r/2, 4+r/2)$.

This point Q lies on the plane $2x - y + z + 3 = 0$

$\therefore 2(r+1) - (3-r/2) + (4+r/2) + 3 = 0$

$\therefore 3r = -6$. Hence $r = -2$.

Therefore P is $(1-4, 3+2, 4-2)$ That is, P' is $(-3,5,2)$.

Therefore the image of $(1,3,4)$ under reflection is $(-3,5,2)$

We now find the point where the line $\frac{x-1}{1} = \frac{y-3}{-2} = \frac{z-4}{-3} \quad \cdots (2)$

meets the plane $2x - y + z + 3 = 0$.

Any point on the line (2) is $R(r+1, 3-2r, 4-3r)$ and it lies in the plane (1).

Hence $2(1+2r) - (3-2r) + (4-3r) + 3 = 0$.

Therefore $r = -6$. Hence R is $(-5, 15, 22)$.

Since R lies in the plane (1) the image of R in the plane (1) is itself.

Therefore the line RP' is the image of the line (2) and its equation are

$\frac{x+3}{-3+5} = \frac{y-5}{5-15} = \frac{z-2}{2-22}$.

That is, $\frac{x+3}{2} = \frac{y-5}{-10} = \frac{z-2}{-20}$

That is, $\frac{x+3}{1} = \frac{y-5}{-5} = \frac{z-2}{-10}$

Exercises 2.1.15. 1. Find the equation of the straight line joining the points $(2, 5, 8)$ and $(-1, 6, 3)$.

2. Find the perpendicular distance from $P(3, 9, -1)$ to the line $\frac{x+8}{-8} = \frac{y-31}{1} = \frac{z-13}{5}$.

3. Put in the symmetrical form the lines

   (i) $3x - 2y + z - 1 = 0 = 5x + 4y - 6z - 2$.

   (ii) $4x + 4y - 5z - 12 = 0 = 8x + 12y - 13z - 32$.

2.2 PLANE AND A STRAIGHT LINE

Theorem 2.2.1. *Let* $L : \frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ $\cdots$ *(1)*

be a line and $\pi : ax + by + cz + d = 0$ $\cdots$ *(2)*

be a plane. The condition that

(i) the line L *be parallel to the plane* π *is* $ax_1 + by_1 + cz_1 + d \neq 0$ *and* $al + bm + cn = 0$*;*

(ii) the line L *to lie in the plane* π *is* $ax_1 + by_1 + cz_1 + d = 0$ *and* $al + bm + cn = 0$.

Proof. [**i**] The coordinates of any point on the line (1) are $(x_1 + lr, y_1 + mr, z_1 + nr)$.

Suppose this point lies on the plane (2), then

$a(x_1 + lr) + b(y_1 + mr) + c(z_1 + nr) + d = 0$

That is, $r(al + bm + cn) + (ax_1 + by_1 + cz_1 + d) = 0$ $\cdots(3)$

If the line is parallel to plane, no point of the line lies on the plane.

Therefore no value of r satisfies(3). Hence $al + bm + cn = 0$ and

$ax_1 + by_1 + cz_1 + d \neq 0$.

[**ii**] The line (1) will lie in the plane (2) if every point on the line lies in the plane.

Hence (3) must be satisfied by all values of r.

Therefore $al + bm + cn = 0$ and $ax_1 + by_1 + cz_1 + d = 0$. □

Example 2.2.2. Find the equations of the orthogonal projection of the line $\frac{x-2}{4} = \frac{y-1}{2} = \frac{z-4}{3}$ on to the plane $8x + 2y + 9z - 1 = 0$.

The required orthogonal projection lies in the plane drawn through the given line perpendicular to the given plane.

The equation of any plane containing the given line is

$$A(x-2) + B(y-1) + C(z-4) = 0 \qquad \cdots (1)$$

subject to the condition

$$4A + 2B + 3C = 0 \qquad \cdots (2)$$

Plane (1) is perpendicular to the plane $8x + 2y + 9z - 1 = 0$

Therefore $8A + 2B + 9C = 0$ $\cdots (3)$

From (2) and (3), we get $\frac{A}{12} = \frac{B}{-12} = \frac{C}{-8}$. That is, $\frac{A}{3} = \frac{B}{-3} = \frac{C}{-2}$

Substituting the value of A, B, C in (1), we get the equation of the plane(1) as

$3(x-2) - 3(y-1) - 2(z-4) = 0$

That is, $3x - 3y - 2z + 5 = 0$.

2.3 Coplanar Lines

Theorem 2.3.1. *The condition for two lines* $\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ *$\cdots$ (1)*

and $\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$ *$\cdots$ (2)*

to be coplanar is $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$

Proof. Two coplanar lines must be either parallel or intersecting.

The lines (1) and (2) are parallel if $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$.

Suppose the lines are not parallel.

Therefore Let the lines intersect at P(say).

Therefore the coordinates of any point on the line (1) are

$(x_1 + l_1 r, y_1 + m_1 r, z_1 + n_1 r)$.

The coordinates of any point on the line (2) are $(x_2 + l_2 r_1, y_2 + m_2 r_1, z_2 + n_2 r_1)$

Since P is a common point for (1) and (2), we have

$x_1 + l_1 r = x_2 + l_2 r_1; y_1 + m_1 r = y_2 + m_2 r_1; z_1 + n_1 r = z_2 + n_2 r_1$ for some values of r and r_1.

$\therefore \quad (x_1 - x_2) + l_1 r - l_2 r_1 = 0$

$(y_1 - y_2) + m_1 r - m_2 r_1 = 0$

$(z_1 - z_2) + n_1 r - n_2 r_1 = 0$

Eliminating r and r_1 form the above three equations, we get

$$\begin{vmatrix} x_1 - x_2 & l_1 & l_2 \\ y_1 - y_2 & m_1 & m_2 \\ z_1 - z_2 & n_1 & n_2 \end{vmatrix} = 0$$

That is, $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$

Further the above conditions is satisfied even if $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$ and hence it is the required condition for the given lines to be coplanar. Hence the theorem. □

Note 2.3.2. If the line given by (1) lies in the plane

$ax + by + cz + d = 0$ $\cdots(1)$

then, we have $ax_1 + by_1 + cz_1 + d = 0$ $\cdots(2)$

and $al + bm + cn = 0$

From(1) and (2), the equation of the plane of the plane containing the line can be written as $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ together with the condition $al + bm + cn = 0$.

Theorem 2.3.3. *The angle between the line* $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ *and the plane* $ax + by + cz = 0$ *is given by* $\sin\theta = \frac{al+bm+cn}{\sqrt{(a^2+b^2+c^2)}\sqrt{(l^2+m^2+n^2)}}$.

Proof. Let θ be the angle between the given plane and the straight line.

Therefore $90 - \theta$ is the angle between the line and the normal to plane.

The direction ratios of the normal to the plane are a, b, c and the direction ratios of the line are l, m, n.

Therefore $\cos(90 - \theta) = \frac{al+bm+cn}{\sqrt{(a^2+b^2+c^2)}\sqrt{(l^2+m^2+n^2)}}$.

Hence the theorem. □

Corollary 2.3.4. *The line* $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ *is parallel to the plane* $ax+by+cz+d=0$ *if and only if* $al+bm+cn=0$.

Theorem 2.3.5. *The equation of the plane containing two lines*

$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ $\cdots$ *(1)*

and $\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$ $\cdots$ *(2)*

is
$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

Proof. The equation of the plane containing the line (1) is

$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$ $\cdots$(A)

subject to $al_1+bm_1+cn_1=0$ $\cdots$(B)

Also the line (2) is perpendicular to normal to the plane (A).

$\therefore al_2+bm_2+cn_2=0$ $\cdots$(C)

Eliminating a,b,c from (A),(B) and(C), we get the required equation. □

Theorem 2.3.6. *The length of the perpendicular from a point* $P(x_1,y_1,z_1)$ *to the line*

$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ $\cdots$ *(1)*

is $\left[(x_1-\alpha)^2+(y_1-\beta)^2+(z_1-\gamma)^2 - \frac{[l(x_1-\alpha)+m(y_1-\beta)+n(z_1-\gamma)^2}{l^2+m^2+n^2}\right]^{1/2}$

Proof. Let PL be the perpendicular from P to the line (1).

Then L is $(\alpha+lr, \beta+mr, \gamma+nr)$ for some r.

Therefore d.r of PL are $x_1-\alpha-lr, y_1-\beta-mr, z_1-\gamma-nr$.

Since PL is perpendicular to the line (1), we have

$l(x_1-\alpha-lr)+m(y_1-\beta-mr)+n(z_1-\gamma-nr)=0$

$\therefore l(x_1-\alpha)+m(y_1-\beta)+n(z_1-\gamma)=r(l^2+m^2+n^2)$ $\cdots$(2)

Therefore $r = \frac{l(x_1-\alpha)+m(y_1-\beta)+n(z_1-\gamma)}{(l^2+m^2+n^2)}$

$$\begin{aligned}
\text{Now } PL^2 &= (x_1-\alpha-lr)^2+(y_1-\beta-mr)^2+(z_1-\gamma-nr)^2 \\
&= (x_1-\alpha)^2+(y_1-\beta)^2+(z_1-\gamma)^2-2r[l(x_1-\alpha)+m(y_1-\beta)+ \\
&\qquad n(z_1-\gamma)]+r^2(l^2+m^2+n^2) \\
&= (x_1-\alpha)^2+(y_1-\beta)^2+(z_1-\gamma)^2-2r^2(l^2+m^2+n^2)+ \\
&\qquad r^2(l^2+m^2+n^2) \\
&= (x_1-\alpha)^2+(y_1-\beta)^2+(z_1-\gamma)^2-r^2(l^2+m^2+n^2)
\end{aligned}$$

Hence the result. □

2.4 Skew lines

Definition 2.4.1. *Two straight lines in space which are not coplanar are called skew lines.*

Note 2.4.2. *There is only one straight line which is perpendicular to both the skew lines.*

Definition 2.4.3. *The shortest distance (abbreviated by S.D.) between two skew lines is the length of the common perpendicular drawn to the lines.*

Theorem 2.4.4. *Shortest distance between the skew lines*

$$L_1 : \frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1} \qquad \cdots (1)$$

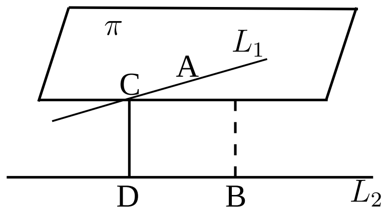
$$\textit{and } L_2 : \frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2} \qquad \cdots (2)$$

is given by $S.D = \dfrac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}}{\sqrt{\sum (l_1m_2 - l_2m_1)^2}}$

Proof. Let $A(x_1, y_1, z_1)$ be a point on the line L_1 and $B(x_2, y_2, z_2)$ be a point on L_2. Consider a plane π containing the line L_1 and parallel to L_2.

Then every point on L_2 will be equidistant from this plane.

Hence the shortest distance is the perpendicular distance of any point on L_2 to the plane π.



Therefore the shortest distance is CD (refer figure). The equation of the plane π can be taken as

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ $\cdots(3)$

subject to $al_1 + bm_1 + cn_1 = 0$ $\cdots(4)$

Since the plane is parallel to L_2 we have $al_2 + bm_2 + cn_2 = 0$ $\cdots(5)$

From (4)and(5) we get $\frac{a}{m_1n_2 - m_2n_1} = \frac{b}{n_1l_2 - n_2l_1} = \frac{c}{l_1m_2 - l_2m_1}$.

Therefore the equation of the plane π is

$(m_1n_2 - m_2n_1)(x - x_1) + (n_1l_2 - n_2l_1)(y - y_1) + (l_1m_2 - l_2m_1)(z - z_1) = 0$ $\cdots(6)$

Therefore the shortest distance $S.D =$ the perpendicular distance from $B(x_2, y_2, z_2)$ to the plane(6).

$$\therefore \quad S.D = \frac{(m_1n_2 - m_2n_1)(x_2 - x_1) + (n_1l_2 - n_2l_1)(y_2 - y_1) + (l_1m_2 - l_2m_1)(z_2 - z_1)}{\sqrt{\sum (m_1n_2 - m_2n_1)^2}}$$

Hence the shortest distance can be expressed in the form as given in the theorem.

Now, we find the equation of the line of shortest distance CD.

CD is the intersection of the planes ACD and BDC.

Let l, m, n be the direction ratios of CD.

Therefore the equation of the plane ACD is

$$\pi_1 : \begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 \qquad \cdots (1)$$

The equation of the plane BDC is

$$\pi_2 : \begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix} = 0 \qquad \cdots (2)$$

Since CD is perpendicular to L_1 and L_2, we have

$ll_1 + mm_1 + nn_1 = 0$ and $ll_2 + mm_2 + nn_2 = 0$

Hence $\frac{l}{m_1n_2 - m_2n_1} = \frac{m}{n_1l_2 - n_2l_1} = \frac{n}{l_1m_2 - l_2m_1}$ $\cdots (3)$

Therefore the equations of the shortest distance CD are given by $\pi_1 = 0$ and $\pi_2 = 0$ where l, m, n are given by (3). □

Note 2.4.5. *If the shortest distance between the two lines L_1 and L_2 is zero then the lines are coplanar.*

2.4.1 Solved problems

Problem 2.4.6. *Find the equation of the plane containing the point* $(-1, 7, 2)$ *and the line* $\frac{x+3}{2} = \frac{y+2}{3} = \frac{z-2}{-2}$ $\cdots$ *(1)*

Solution. The equation of the plane containing the line (1) is

$a(x+3) + b(y+2) + c(z-2) = 0$ $\cdots (2)$

subject to $2a + 3b - 2c = 0$ $\cdots (3)$

Since the plane passes through $(-1, 7, 2)$, we have from (2)

$2a + 9b = 0$ $\cdots (4)$

From (3) and (4), we have $\frac{a}{18} = \frac{b}{-4} = \frac{a}{24}$

Therefore $a = 18k, b = -4k, c = 24k$;

Substituting in (2), we get the equation of the plane as $9x - 2y + 12z = 1$.

Problem 2.4.7. *Find the equation of the plane which contains the two parallel lines*

$$\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3} \qquad \cdots(1)$$

$$\frac{x-3}{1}=\frac{y+2}{2}=\frac{z+4}{3} \qquad \cdots(2)$$

Solution. The equation of the plane containing the line (1) is given by

$a(x-1)+b(y-2)+c(z-3)=0$ $\cdots(3)$

subject to $a+2b+3c=0$ $\cdots(4)$

Since the line (2) lies on the plane (3), the point $(3,-2,-4)$ lies on it.

Therefore from (3), we have $2a-4b-7c=0$ $\cdots(5)$

From (4) and (5), we have $\frac{a}{-2}=\frac{b}{13}=\frac{c}{-8}$.

Therefore $a=-2k, b=13k, c=-8k$.

Therefore the equation of the plane is $-2x+13y-8z=0$.

Problem 2.4.8. *Prove that the lines* $\frac{x}{l}=\frac{y}{m}=\frac{z}{n};\frac{x}{l_1}=\frac{y}{m_1}=\frac{z}{n_1};\frac{x}{l_2}=\frac{y}{m_2}=\frac{z}{n_2}$ *are coplanar if* $\begin{vmatrix} l & m & n \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0.$

Solution. Obviously, the three lines intersect at $(0,0,0)$. Hence they determine a plane.

Now, the equation of the plane containing these two lines is

$$\begin{vmatrix} x & y & z \\ l & m & n \\ l_1 & m_1 & n_1 \end{vmatrix}=0$$

That is, $(mn_1-m_1n)x-(ln_1-l_1n)y+(lm_1-l_1m)z=0$ $\cdots(1)$

Suppose the three lines are coplanar. Then the third line also lies on (1).

Therefore the normal to the plane (1) is perpendicular to the third line.

Therefore $l_2(mn_1-m_1n)-m_2(ln_1-l_1n)+n_2(lm_1-l_1m)=0$

That is, $\begin{vmatrix} l_2 & m_2 & n_2 \\ l & m & n \\ l_1 & m_1 & n_1 \end{vmatrix}=0$

That is, $\begin{vmatrix} l & m & n \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$

Problem 2.4.9. *Show that the lines* $\frac{x-1}{1} = \frac{y-4}{2} = \frac{z-5}{2}$ *and* $\frac{x-5}{2} = \frac{y-8}{3} = \frac{z-7}{2}$ *are coplanar and find the equation of the plane containing them.*

Solution. Here $(x_1, y_1, z_1) = (2, 4, 5)$ and $(x_2, y_2, z_2) = (5, 8, 7)$.
We know that the lines are coplanar if

$$\begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ l & m & n \\ l_1 & m_1 & n_1 \end{vmatrix} = 0$$

Now, $\begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ l & m & n \\ l_1 & m_1 & n_1 \end{vmatrix} = \begin{vmatrix} -3 & -4 & -2 \\ 1 & 2 & 2 \\ 2 & 3 & 2 \end{vmatrix} = 0$ (verify)

Equation of the plane containing the line is $\begin{vmatrix} x-2 & y-4 & z-5 \\ 1 & 2 & 2 \\ 2 & 3 & 2 \end{vmatrix} = 0$

That is, $2x - 2y + z - 1 = 0$

Problem 2.4.10. *Prove that the lines*

$$\frac{x-3}{2} = \frac{y-2}{-5} = \frac{z-1}{3} \quad \cdots (1)$$

$$\textit{and} \quad \frac{x-1}{-4} = y+2 = \frac{z-6}{2} \quad \cdots (2)$$

are coplanar. Find the point of intersection. Also find the equation of the plane determined by the lines.

Solution. Here the condition $\frac{l}{l_1} = \frac{m}{m_1} = \frac{n}{n_1}$ is not satisfied. Hence the lines are not parallel. Hence if the lines are to be coplanar they must intersect.
The coordinates of any point on the line (1) are

$$P(2r+3, -5r+2, 3r+1) \quad \cdots (3)$$

The coordinates of any point on the line (2) are

$Q(-4r_1+1, r_1-2, 2r_1+6)$ $\cdots(4)$

The two lines intersect if $2r+3=-4r_1+1$;

$-5r+2=r_1-2$ and $3r+1=2r_1+6$ for some values of r, r_1.

Therefore $2r+4r_1=-2$ $\cdots(5)$

$-5r-r_1=-4$ $\cdots(6)$

$3r-2r_1=5$ $\cdots(7)$

Solving (5) and (6), we get $r=1$ and $r_1=-1$.

These values satisfy the equation (7) also.

Hence the two lines intersect. The point of intersection P is $(5,-3,4)$ (from (3)).

The equation of the plane containing the two lines (1) and (2) is given by

$$\begin{vmatrix} x-3 & y-2 & z-1 \\ 2 & -5 & 3 \\ -4 & 1 & 2 \end{vmatrix} = 0$$

That is, $13x+16y+18z-89=0$.

Problem 2.4.11. *Find the shortest distance and the equation of the line of shortest distance between the straight lines* $\frac{x+3}{-4}=\frac{y-6}{6}=\frac{z}{2}$ *and* $\frac{x+2}{-4}=\frac{y}{1}=\frac{z-7}{1}$

Solution.

$$S.D = \frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}}{\sqrt{\sum (m_1n_2-m_2n_1)^2}}$$

Here $(x_1,y_1,z_1)=(-3,6,0)$;$(x_2,y_2,z_2)=(-2,0,7)$

$(l_1,m_1,n_1)=(-4,6,2)$ and $(l_2,m_2,n_2)=(-4,1,1)$.

Now, $\begin{vmatrix} x_1-x_2 & y_1-y_2 & z_1-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = \begin{vmatrix} 1 & -6 & 7 \\ -4 & 6 & 2 \\ -4 & 1 & 1 \end{vmatrix} = 168$

$\sum (m_1n_2-m_2n_1)^2 = (m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2+(l_1m_2-l_2m_1)^2$

$=(6-2)^2+(-8+4)^2+(-4+24)^2$

$=16+16+400+=432$

$\therefore \quad S.D=\frac{168}{\sqrt{432}}=\frac{168}{12\sqrt{3}}=\frac{14}{\sqrt{3}}$

The equation of the line of shortest distance is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 = \begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix}$$

$$\text{where } \frac{l}{m_1n_2 - m_2n_1} = \frac{m}{n_1l_2 - n_2l_1} = \frac{n}{l_1m_2 - l_2m_1}$$

$$\therefore \frac{l}{4} = \frac{m}{-4} = \frac{n}{20}$$

That is,

$$\frac{l}{1} = \frac{m}{-1} = \frac{n}{5}$$

Hence the equation of shortest lines is

$$\begin{vmatrix} x+3 & y-6 & z- \\ -4 & 6 & 2 \\ 1 & -1 & 5 \end{vmatrix} = 0 = \begin{vmatrix} x-2 & y & z-7 \\ -4 & 1 & 1 \\ 1 & -1 & 5 \end{vmatrix}$$

Therefore $16x + 11y - z - 18 = 0 = 2x + 7y + z - 3$ (verify)

Problem 2.4.12. *Find the shortest distance between the lines L_1 and L_2 if*

$L_1 : \frac{x-3}{3} = \frac{y-6}{-4} = z - 9$

$L_2 : 2x - 2y + z - 3 = 0 = 2x - y + 2z - 9.$

Solution. Let π be the plane through L_2 and parallel to L_1. Then the shortest distance between L_1 and L_2 is the perpendicular distance from $(5, 6, 9)$ to the plane π. The equation of π is of the form.

$2x - 2y + z - 3 + \lambda(2x - y + 2z - 9) = 0$

That is, $(2 + 2\lambda)x + (-2 - \lambda)y + (1 + 2\lambda)z - (3 + 9\lambda) = 0$ $\cdots(1)$

Since π is parallel to L_1, the normal to π is perpendicular to L_1.

Therefore $3(2 + 2\lambda) - 4(-2 - \lambda) + (1 + 2\lambda) = 0$. Hence λ=-$\frac{5}{4}$

Therefore from (1), the equation of π is $2x + 3y + 6z = 33 = 0$.

$$\therefore \quad S.D = \frac{2(5) + 3(6) + 6(9) - 33}{\sqrt{(2^2 + 3^2 + 6^2)}} = 7$$

Problem 2.4.13. *Find the shortest distance between the lines*

$2x - 2y + 3z - 12 = 0 = 2x + 2y + z$ $\cdots$ *(1)*

$2x - z = 05x - 2y + 9$ $\cdots$ *(2)*

Solution. The equation of a plane containing the line (1) is

$2x - 2y + 3z - 12 + \lambda(2x + 2y + z) = 0$

That is, $(2 + 2\lambda)x + (-2 + 2\lambda)y + (3 + \lambda)z - 12 = 0$ $\cdots(3)$

The equation of a plane containing the line (2) is

(2x-z)+$\mu(5x - 2y + 9) = 0$

We find the values of λ and μ such that the planes (3) and (4) are parallel. We have

$\frac{2+2\lambda}{2+5\mu} = \frac{-2+2\lambda}{-2\mu} = \frac{3+\lambda}{-1}$ $\cdots(5)$

Taking the first two ratios in (5), we get

$2\lambda - 3\mu + 7\lambda\mu - 2 = 0$ $\cdots(6)$

Taking the second and third ratios in [5], we get

$\lambda - 3\mu + \lambda\mu + 1 = 0$ $\cdots(7)$

From (7), we get $\lambda = \frac{1+3\mu}{1-\mu}$ $\cdots(8)$

Substituting (8) in (6) we get $2\left(\frac{1+3\mu}{1-\mu}\right) - 3\mu + 7\left(\frac{1+3\mu}{1-\mu}\right)\mu - 2 = 0$

That is, $2 + 6\mu - 3\mu + 3\mu^2 + 7\mu + 21\mu^2 - 2 + 2\mu = 0$

That is, $24\mu^2 + 12\mu = 0$. Hence $12\mu(2\mu + 1) = 0$

Therefore $\mu = 0$ or $\mu = -\frac{1}{2}$

Hence from (8), we get $\Lambda = 1$ or $\lambda = -\frac{1}{3}$

When $\lambda = 1$,$\mu = 0$ does not satisfy (5).

Hence we take $\lambda = -\frac{1}{3}$ and $\mu = -\frac{1}{2}$

Hence the planes (3) and (4) become

$x - 2y + 2z - 9 = 0$ $\cdots(9)$

$x - 2y + 2z + 9 = 0$ $\cdots(10)$

The point of intersection of (9) with the z-axis is $(0, 0, 9/2)$

Therefore the required shortest distance is the perpendicular distance from $(0, 0, 9/2)$ to the plane (10) $= \pm\left(\frac{18}{\sqrt{[1^2+(-2)^2+2^2]}}\right) = 6$

Problem 2.4.14. *Find the shortest distance and the equation of the line of shortest distance in symmetrical form of the lines*

$\frac{x-3}{3} = \frac{y+9}{-16} = \frac{z-10}{7}$ *and* $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$.

Solution. Coordinates of any point P on the first line are $(3r+8, -16r-9, 7r+10)$.

Coordinates of any point q on the second line are $(3s+15, 8s+29, -5s+5)$.

Let PQ be the shortes distance.

Direction ratios of PQ are $(3s-3r+7, 8s+16r+38)+7(-5s-7r-5)=0$

and $3(3s-3r+7)-16(8s+16r+38)-5(-5s-7r-5)=0$.

That is, $77s+157r=-311$ $\cdots(1)$

and $7s+11r=-25$ $\cdots(2)$

Solving (1) and (2) we get *r=-1* and *s=-2*.

Therefore P is (5,7,3) and Q is (9,13,15).

$\therefore PQ = \sqrt{(9-5)^2+(13-7)^2+(15-3)^2} = \sqrt{16+36+144}$

$\sqrt{196} = 14.$

The equations of the line of shortest distance are

$\frac{x-5}{5-9} = \frac{y-7}{7-13} = \frac{z-3}{3-15}$

That is, $\frac{x-5}{-4} = \frac{y-7}{-6} = \frac{z-3}{-12}$

That is, $\frac{x-5}{2} = \frac{y-7}{3} = \frac{z-3}{6}$

Problem 2.4.15. *Find the distance of the point* $(3,4,5)$ *from the point of intersection of* $\frac{x-3}{1} = \frac{y-4}{2} = \frac{z-5}{2}$ *with the plane* $x+y+z=2$.

Solution. We note the point $A(2,3,5)$ is a point on the line.

Any point on the line is $P(r+3, 2r+4, 2r+5)$.

If P is the point of intersection of the line with the plane, then P lies on the plane.

$\therefore \quad (r+3)+(2r+4)+(2r+5)=2$

$\therefore \quad 5r=-10$. Hence $r=-2$.

Therefore P is $(1,0,1)$.

Hence the required distance $AP = \sqrt{(3-1)^2+(4-0)^2+(5-1)^2}$

$=\sqrt{36}$

$=6$

Problem 2.4.16. *Find the shortest distance between the lines* $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ *and* $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$.

Solution.

$$S.D = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}}{\sqrt{\sum (l_1 m_2 - l_2 m_1)^2}}$$

Here $(x_1, y_1, z_1) = (1, 2, 3)$ and $(x_2, y_2, z_2) = (2, 3, 4)$

Now, $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$

Hence the lines are coplanar.

Problem 2.4.17. *Find in symmetrical form, the equation of the orthogonal projection of the line* $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-4}{4}$ $\cdots$ *(1)*

on the plane $3x + 4y + 5z = 0$. $\cdots$ *(2)*

Solution. The orthogonal projection of the line (1) is the intersection of plane (2) and plane containing the line (1) perpendicular to the plane (2).

The equation of the plane containing the line (1) is

$a(x - 1) + b(y - 2) + C(z - 4) = 0$ $\cdots (3)$

subject to $2a + 3b + 4c = 0$ $\cdots (4)$

The plane (3) is perpendicular to (2). Hence $3a + 4b + 5c = 0$.

From (4) and (5), we have $\frac{a}{-1} = \frac{b}{2} = \frac{c}{-1} = k$ (say)

$a = -k; b = 2k; c = -k.$

Therefore the equation of the plane (3) is $x - 2y + z - 1 = 0$. $\cdots (6)$

Therefore the required line is the intersection of the plane (2) and (6).

Now, we get the equations of the line in the symmetrical form.

Let α, β, γ be the direction ratios of the line.

$\therefore 3\alpha + 4\beta + 5\gamma = 0$ and $\alpha - 2\beta + \gamma = 0$

$\therefore \frac{\alpha}{14} = \frac{\beta}{2} = \frac{\gamma}{-10}$

Hence the direction ratios are $7, 1, -5$.

The line meets the xy plane $a = 0$. Hence $x - 2y = 1$ and $3x + 4y = 0$.

Solving the two equations, we get the point as $(\frac{2}{5}, \frac{-3}{10}, 0)$.

Therefore the equations of the line are $\frac{x-\frac{2}{5}}{7} = \frac{y+\frac{3}{10}}{1} = \frac{z}{-5}$

Problem 2.4.18. *Find the condition that the lines*

$a_1x + b_1y + c_1z + d_1 = 0 = a_2x + b_2y + c_2z + d_2$ $\cdots$ *(1)*

and $a_3x + b_3y + c_3z + d_3 = 0 = a_4x + b_4y + c_4z + d_4$ $\cdots$ *(2)*

may be coplanar.

Solution. Let the lines represented by the equations (1) and (2) be coplanar. Then (1) and (2) will have a common point say (x_1, y_1, z_1).

Hence it lies on all the four planes which determine the two lines.

$$\begin{aligned} \therefore \quad a_1x_1 + b_1y_1 + c_1z_1 + d_1 &= 0 \\ a_2x_1 + b_2y_1 + c_2z_1 + d_2 &= 0 \\ a_3x_1 + b_3y_1 + c_3z_1 + d_3 &= 0 \\ a_4x_1 + b_4y_1 + c_4z_1 + d_4 &= 0 \end{aligned}$$

Eliminating x_1, y_1, z_1 from the above four equations, we get the required conditions as

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0$$

Exercises 2.4.19.

1. Prove that the following lines are coplanar and find the equation of the plane in which they lie.

   (i) $\frac{x-3}{2} = \frac{y-2}{-5} = \frac{z-1}{3}$; $\frac{x-1}{-4} = \frac{y+2}{1} = \frac{z-6}{2}$;

   (ii) $x + 1 = \frac{y+2}{2} = z - 1$; $x - 2y + 2z - 3 = 0 = x - 4y + 5z - 8$.

2. Show that the lines $\frac{x-8}{3} = \frac{y+9}{-16} = \frac{z-10}{7}$ and $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$ are skew lines.

3. Find the shortest distance and the equation of the shortest distance between the following skew lines

   $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$; $\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$.

2.5 Sphere

Definition 2.5.1. *A* ***sphere*** *is the locus of a point in space which moves such that its distance from a fixed point is constant. The fixed point is called the centre of the sphere and the fixed distance is called the radius of the sphere.*

We now proceed to find several forms of the equation of a sphere.

1. Centre radius form

Theorem 2.5.2. *The equation of the sphere with centre $C(a, b, c)$ and radius r is given by $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$.*

Proof. Let $P(x_0, y_0, z_0)$ be any point on the sphere.
Hence $CP^2 = r^2$
Therefore $(x_0 - a)^2 + (y_0 - b)^2 + (z_0 - c)^2 = r^2$.
Therefore the locus of (x_0, y_0, z_0) is $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$. □

Corollary 2.5.3. *The equation of the sphere with centre origin and radius r is $x^2 + y^2 + z^2 = r^2$.*

2. General form of a sphere

Theorem 2.5.4. *The equation $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ represents a sphere with centre $(-u, -v, -w)$ and radius $\sqrt{u^2 + v^2 + w^2 - d}$.*

Proof. The given equation can be written as
$(x + u)^2 + (y + v)^2 + (z + w)^2 = u^2 + v^2 + w^2 - d$.
This represents the locus of a point (x, y, z) which moves such that its distance from the point $C(-u, -v, -w)$ is equal to the constant $\sqrt{u^2 + v^2 + w^2 - d}$
Hence the given equation represents a sphere with centre $(-u, -v, -w)$ and radius $\sqrt{u^2 + v^2 + w^2 - d}$ □

Note 2.5.5. The eqution $ax^2 + ay^2 + az^2 + 2ux + 2vy + 2wz + d = 0$ represents a sphere with a centre $\left(-\frac{u}{a}, -\frac{v}{a}, -\frac{w}{a}\right)$ and radius $\sqrt{\left(\frac{u^2}{a^2} + \frac{v^2}{a^2} + \frac{w^2}{a^2} - \frac{d}{a}\right)}$.

Note 2.5.6. The equation $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ can be denoted as $S=0$ where $S \equiv x^2+y^2+z^2+2ux+2vy+2wz+d$.

3. Diameter form

Theorem 2.5.7. *The equation of the sphere described on the line joining the points $A(x_1, y_1, z_1)$ and (x_2, y_2, z_2) as diameter is given by*
$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0.$

Proof. Let $P(x, y)$ be any point on the sphere with AB as diameter.
Therefore the direction ratios of AP are $x-x_1, y-y_1, z-z_1$ and the direction ratios of BP are $x-x_2, y-y_2, z-z_2$.
Consider the circle passing through A, B and P. This circle also has AB as diameter and hence $\angle APB = 90^0$. [i.e] AP is perpendicular to BP.
Therefore $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)$=0
Since this is true for any point (x, y, z) on the sphere it represents the equation of the required sphere. □

2.6 Tangent Plane

Definition 2.6.1. *The straight line joining two points P and Q on a surface is called a chord of the surface. When Q moves along the surface and ultimately coincides with P the limiting position of PQ touches the surface at P and is called a tangent line of the surface.*
In the case of a sphere with centre C there are many tangent lines at a point P on it, all of them being perpendicular to the radius CP. All these tangents lie on the plane through P perpendicular to CP. This plane is called the **tangent plane** *of the sphere at P.*

Theorem 2.6.2. *The equation of the tangent plane to the sphere*
$x^2+y^2+z^2+2ux+2vy+2wz+d=0$ *at* $P(x_1, y_1, z_1)$ *is*
$xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0.$

Proof. The centre of the given sphere is $C(-u,-v,-w)$. The tangent plane to the sphere at $P(x_1,y_1,z_1)$ passes through P and has CP as its normal.
The direction ratios of CP are x_1+u, y_1+v, z_1+w.
Hence the equation of the tangent plane at P is
$(x_1+u)(x-x_1)+(y_1+v)(y-y_1)+(z_1+w)(z-z_1)=0.$
That is, $xx_1+yy_1+zz_1+ux+vy+wz=x_1^2+y_1^2+z_1^2+ux_1+vy_1+wz_1$
That is, $xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d$

$= x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d$

$= 0$ [since the (x_1,y_1,z_1) lies on the sphere]

Hence the result. □

2.6.1 Angle of intersection of two spheres

The angle of intersection of two spheres at a common point is the angle between the tangent planes to them at that point. Since the angle between the two tangent planes at the common point is same as the angle between their normals at that point we note that the angle between the two sphere is same as the angle between the radii of the two spheres at the common point. Also we note that the angle of intersection at every common point of the sphere is the same.
Suppose the two spheres $S=0$ and S_1=0 with centre A and B and radii r and r_1, respectively, cut orthogonally then, $AB^2=AP^2+BP^2$, where P is the common Point and $AP=r$ and $BP=r_1$.

Theorem 2.6.3. *The condition for two spheres*
$S=x^2+y^2+z^2+2ux+2vy+2wz+d=0$ *and*
$S_1=x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$ *to cut orthogonally is*
$2uu_1+2vv_1+2ww_1=d+d_1$.

Proof. The centre of $S=0$ is $A(-u,-v,-w)$ and radius
$r=\sqrt{(u^2+v^2+w^2-d)}$.
The centre of $S=0$ is $B(-u_1,-v_1,-w_1)$ and radius
$r_1=\sqrt{(u_1^2+v_1^2+w_1^2-d_1)}$.

Let P be a common point.

The two spheres cut orthogonally if $r^2 + r_1^2 = AB^2$.

$$\begin{aligned}\therefore\quad (u^2+v^2+w^2-d)+(u_1^2+v_1^2+w_1^2-d_1) &= (u_1-u)^2+(v_1-v)^2+(w_1-w)^2\\ &= u^2+u_1^2-2uu_1+v^2+v_1^2-2vv_1+\\ &\qquad w^2+w_1^2-2ww_1\end{aligned}$$

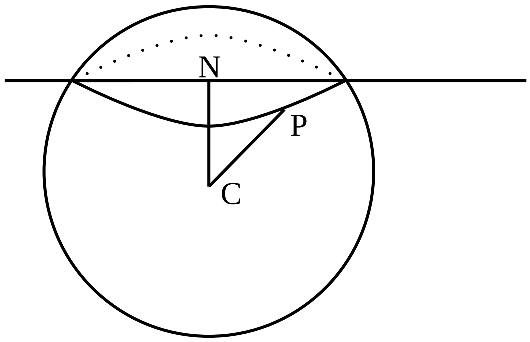
$$\therefore\quad 2uu_1+2vv_1+2ww_1 = d+d_1$$

□

2.7 Plane Section

Theorem 2.7.1. *A plane section of a sphere is a circle.*

Proof. Let a plane π cut a sphere of radius r and Centre C.



Let P be a point on the plane section. We claim that the locus of P is a circle.

Let N be the foot of the perpendicular drawn from C to the plane π.

Therefore NP=$\sqrt{CP^2 - CN^2} = \sqrt{r^2 - CN^2}$ which is a constant. Hence the locus of P is a circle with centre N and radius $\sqrt{r^2 - CN^2}$. □

Note 2.7.2. The section of a sphere by a plane through its centre is known as a great circle and the centre and radius of a great circle are the same as that of the sphere.

Note 2.7.3. The curve of intersection of a sphere by a plane is a circle. Hence a circle can be represented by two equations one being the equation of a sphere S and the other a plane π. Hence $S = 0$ and $\pi = 0$ given together represent a circle.

Note 2.7.4. Let $S = x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ and $\pi = ax + by + cz + d = 0$. Then S+$\lambda\pi = 0$ (where λ is a constant) represents the equation of a sphere passing through the circle given by $S = 0$ and $\pi = 0$. For, $s + \lambda\pi = 0$ represents a sphere. Further $S = \lambda\pi = 0$ is satisfied by the points common to S=0 and $\pi = 0$.

Note 2.7.5. Two intersecting sphere also determine a circle. For, if $S = 0$ and $S_1 = 0$ represent two spheres then $S - S_1 = 0$ is a first degree equation in x, y, z and hence represents a plane.

Hence '$S = 0$ and $S - S_1 = 0$' or '$S_1 = 0$ and $S - S_1 = 0$' determine a circle.

Solved Problems

Problem 2.7.6. *Find the equation of the sphere with centre* $(1, -1, 2)$ *and radius 3.*

Solution. The required equation is $(x - 1)^2 + (y - 1)^2 + (z - 1)^2 = 3^2$

That is, $x^2 + y^2 + z^2 - 2x + 2y - 4z - 3 = 0$.

Problem 2.7.7. *Obtain the equation of the sphere having its centre at the point* $(6, -1, 2)$ *and touching the plane* $2x - y + 2z = 0$.

Solution. Since the plane touches the sphere, the radius r is the perpendicular distance from the centre $(6, -1, 2)$ to the plane $2x - y + 2z - 2 = 0$.

Therefore r=$\pm \left[\frac{2(6)-(-1)+2(2)-2}{\sqrt{[2^2+(-1)^2+2^2]}}\right] = \left[\frac{12+1+4-2}{\sqrt{9}}\right] = \frac{15}{3} = 5$

Therefore the equation of the sphere is $(x - 6)^2 + (y + 1)^2 + (z - 2)^2 = 5^2$.

That is, $x^2 + y^2 + z^2 - 12x + 2y - 4z + 16 = 0$.

Problem 2.7.8. *Find the equation of the sphere passing through the points* $(0, 0, 0)$, $(1, 0, 0)$ $(0, 1, 0)$ *and* $(0, 0, 1)$.

Solution. Let the equation of the sphere be

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

It passes through the origin and so $d = 0$.

The point $(1, 0, 0)$ lies on the sphere and so $1 + 2u + d = 0$ and hence $u = -1/2$.

The point $(0,1,0)$ lies on the sphere and so $1+2v+d=0$ and hence $v=-1/2$.

The point $(0,0,1)$ lies on the sphere and so $1+2w+d=0$ and hence $w=-1/2$.

Therefore the equation of the sphere is $x^2+y^2+z^2-x-y-z=0$.

Problem 2.7.9. *Find the equation of the sphere passing through the points* $(1,1,2)$, $(-1,1,2)$ *and having the centre of the sphere on the line*
$x+y-z-1=0=2x-y+z-2$.

Solution. Let the equation of the sphere be

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$

It passes through the points $(1,1,-2)$ and $(-1,1,2)$.

Therefore $1+1+4+2u+2v-4w+d=0$

Therefore $2u+2v-4w+d=-6$ $\cdots(1)$

Similarly, $-2u+2v+4w+d=-6$ $\cdots(2)$

The centre $(-u,-v,-w)$ lies on the line determined by the two planes

$x+y-z-1=0$ and $2x-y+z-2=0$.

Therefore $-u-v+w=1$ $\cdots(3)$

$-2u+v-w=2$ $\cdots(4)$

(1)-(2) $\Longrightarrow 4u-8w=0$ $\cdots(5)$

From (5), we get $u=2w$.

Therefore (3) and (4) become $-w-v=1$. $\cdots(6)$

$-5w+v=2$ $\cdots(7)$

From (6) and (7), we get $w=-1/2$ and $v=-1/2$.

From (3), we get $u=-1$ and from (1), we get $d=-5$.

Therefore the equation of the sphere becomes $x^2+y^2+z^2-2x-y-z-5=0$.

Problem 2.7.10. *Find the equation of the sphere passing through the circle* $x^2+y^2+z^2-4=0$, $2x+4y+6z-1=0$ *having its centre on the plane* $x+y+z=6$.

Solution. The equation of the sphere passing through the circle determined by the sphere and the plane is given by

$x^2+y^2+z^2-4+\lambda(2x+4y+6z-1)=0$

That is, $x^2+y^2+z^2-4+2\lambda x+4\lambda y+6\lambda z-4-\lambda=0$ $\cdots(1)$

Its centre is $(\lambda, -2\lambda, -3\lambda)$.

This centre lies on the plane $x + y + z = 6$.

Therefore $\lambda - 2\lambda - 3\lambda = 6$. Hence $-6\lambda = 6$. Hence $\lambda = -1$

Equation of the required sphere is got from (1) as

$x^2 + y^2 + z^2 - 4 - 2x - 4y - 6z - 3 = 0$

Problem 2.7.11. *Show that the sphere* $s{=}x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ *will cut the spheres* $S_1 = x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0$ *in a great circle if* $2uu_1 + 2vv_1 + 2ww_1 - (d + d_1) = 2r_1^2$ *where* $2r_1^2$ *where* r_1 *is the radius of the later sphere.*

Solution. The plane π determined by $S = 0$ and $S_1 = 0$ is $S - S_1 = 0$ and it is

$\pi : 2(u - u_1)x + 2(v - v_1)y + 2(w - w_1)z + d - d_1 = 0$.

The intersection of the spheres $S = 0$and $S_1 = 0$ will be a great circle if the centre $(-u_1, -v_1, -w_1)$ of $S_1 = 0$ lies on the plane $\pi = 0$.

$\therefore 2(u - u_1)u_1 + 2(v - v_1)v_1 + 2(w - w_1)w_1 + d - d_1 = 0$.

$\therefore 2uu_1 + 2vv_1 + 2ww_1 + d - d_1 - 2u_1^2 - 2v_1^2 - 2w_1^2 = 0$.

$\therefore 2uu_1 + 2vv_1 + 2ww_1 = 2(u_1^2 + v_1^2 + w_1^2) + d - d_1$

$=2(r_1^2 + d_1) + d - d_1$ (since $r_1^2 = u_1^2 + v_1^2 + w_1^2 - d_1$)

$=2r_1^2 + d + d_1$

$\therefore 2uu_1 + 2vv_1 + 2ww_1 - (d + d_1) = 2r_1^2$

Problem 2.7.12. *Prove that the two spheres*

$S_1 = x^2 + y^2 + z^2 - 2x + 4y - 4z = 0; S_2 = x^2 + y^2 + z^2 + 10x + 2z + 10 = 0$

touch each other and find the point of contact.

Solution. The centre of $S_1 = 0$ is $C_1(1, -2, 2)$ and radius is

$r_1 = \sqrt{(-1)^2 + 2^2 + (-2)^2} = 3$

The centre of $S_2{=}0$ is $C_2(-5, 0, -1)$ and radius is

$r_2 = \sqrt{5^2 + 1^2 - 10} = \sqrt{16} = 4$

The distance between the centres, $C_1C_2 = \sqrt{36 + 4 + 9} = 7 = r_1 + r_2$

Hence the two spheres touch each other externally.

The point of contact P is the point of division of the line joining C_1 and C_2 in the ratios 3:4 internally.

Hence P$=\left(\frac{3(-5)+4(1)}{3+4}, \frac{3(0)+4(-2)}{3+4}, \frac{3(-1)+4(2)}{3+4}\right) = \left(-\frac{11}{7}, -\frac{8}{7}, \frac{5}{7}\right)$.

Problem 2.7.13. *Find the equations of tangent planes of the sphere* $x^2 + y^2 + z^2 - 4x - 4y - 4z + 10 = 0$ *which are parallel to the plane* $x - z = 0$.

Solution. Let (x_1, y_1, z_1) be the point on the sphere at which the tangent plane is drawn. The equation of the tangent plane at (x_1, y_1, z_1) is

$xx_1 + yy_1 + zz_1 - 2(x + x_1) - 2(y + y_1) - 2(z + z_1) + 10 - 0$

That is, $(x_1 - 2)x + (y_1 - 2)y + (z_1 - 2)z - 2x_1 - 2y_1 - 2z_1 + 10 = 0$

That is, $(x_1 - 2)x + (y_1 - 2)y + (z_1 - 2)z - 2x_1 - 2y_1 - 2z_1 + 10 = 0 \qquad \cdots(1)$

This line parallel to $x - z = 0$.

$\therefore \frac{x_1-2}{1} = \frac{y_1-2}{0} = \frac{z_1-2}{-1} = k(say)$

$\therefore x_1 = k + 2, y_1 = 2, z_1 = -k + 2 \qquad \cdots(2)$

Since (x_1, y_1, z_1) lies on the sphere, we have

$(k + 2)^2 + 2^2 + (-k + 2)^2 - 4(k + 2) - 8 - 4(-k + 2) + 10 = 0.$

$\therefore k^2 + 4k + 4 + 4 + k^2 - 4k + 4 - 4k - 8 - 8 + 4k - 8 + 10 = 0$

$\therefore 2k^2 - 2 = 0$. Hence $k = \pm 1$.

Therefore from (2), the points are $(3, 2, 1)$ and $(1, 2, 3)$.

Therefore from(1), the equation of the tangent planes are $x - z - 2 = 0$ and $-x + z - 2 = 0$.

Problem 2.7.14. *Prove that the two spheres* $x^2 + y^2 + z^2 + 6y + 2z + 8 = 0$ *and* $x^2 + y^2 + z^2 + 6x + 8y + 4z + 20 = 0$ *intersect each other orthogonally.*

Solution. From the equation of the spheres, we have

$u = 0, v = 3, w = 1, d = 8.$ $u_1 = 3, v_1 = 4, w_1 = 2, d_1 = 20$

Here $2uu_1 + 2vv_1 + 2ww_1 - (d + d_1) = 0 + 24 + 4 - (8 + 20) = 0.$

Hence the two spheres intersect orthogonally.

Problem 2.7.15. *Find the equation of the spheres that passes through the two points* $(0, 3, 0), (-2, -1, -4)$ *and cuts orthogonally the two spheres* $S \ : \ x^2 + y^2 + z^2 + x - 3Z = 2 = 0, S_1 \ : \ 2(x^2 + y^2 + z^2) + x + 3y + 4 = 0.$

Solution. Let the equation of the sphere be

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0.$ $\cdots(1)$

$(0, 3, 0)$ lies on the sphere gives $6v + d = 0.$ $\cdots(2)$

$(-2, -1, -4)$ lies on the sphere gives $-4u - 2v - 8w + d = -21.$ $\cdots(3)$

The sphere S=0 is cut orthogonally by the sphere given by (1)

gives $2u(1/2) + 2v(0) + 2w(-3/2) = d - 2.$

That is, $u - 6w - d = -2$ $\cdots(4)$

The sphere $S_1 = 0$ is cut orthogonally by the sphere given by (1) gives

$2u(1/4) + 2v(3/4) + 2w(0) = d + 4$

That is, $u + 3v - 2d = 4$ $\cdots(5)$

solving (2), (3), (4), (5), we get $u = 1, v = -1, w = 2$ and $d = -3$.

Hence the equation of the sphere is

$x^2 + y^2 + z^2 + 2x - 2y - 4z - 3 = 0.$

Problem 2.7.16. *Find the condition for the plane $lx + my + nz = p$ to be a tangent plane to the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$.*

Solution. The centre of the sphere $S = 0$ is $(-u, -v, -w)$ and the radius is $\sqrt{u^2 + v^2 + w^2 - d}$.

The plane $lx + my + nz = p$is a tangent to the sphere if the perpendicular distance from the centre $(-u, -v, -w)$ = radius.

$\therefore \frac{-lv-mv-nw-p}{\sqrt{l^2+m^2+n^2}} = \sqrt{u^2 + v^2 + w^2 - d}$

$\therefore (lu + mv + nw + p)^2 = (l^2 + m^2 + n^2)(u^2 + v^2 + w^2 - d)$, which is the required condition.

Problem 2.7.17. *Show that the plane $2x - 2y + z + 12 = 0$ touches the sphere $x^2 + y^2 + z^2 - 2x - 4y + 2z - 3 = 0$. Also find the point of contact.*

Solution. The centre of the sphere is $(1, 2, -1)$ and the radius is 3.(verify)

The perpendicular distace from $(1, 2, -1)$ to the given plane

2x-2y+z+12=0 is $\frac{2-4-1+12}{\sqrt{[2^2+(-2)^2+1^2]}} = \frac{9}{3} = 3.$

Thus, the perpendicular distance from the centre to the plane =radius of the sphere.

Therefore $2x - 2y + z + 12 = 0$ is a tangent plane to the given sphere.

The direction ratios of the normal to the plane are 2,-2,1.

Therefore the equation of the perpendicular from the centre to the plane is
$\frac{x-2}{2} = \frac{y-2}{-2} = \frac{z+1}{1}$.

Any point on this line is given by $P(2r+1, -2r+2, r-1)$.

This point P is the point of contact if it lies on the plane $2x - 2y + z + 12 = 0$.

Therefore $2(2r+1) - 2(-2r+2) + (r-1) - 12 = 0$.

Therefore $9r = -9 \Longrightarrow r = -1$.

Therefore the point of contact P is $(-1, 4, -2)$.

Problem 2.7.18. *Find the equation of the sphere through the circle* $x^2 + y^2 + z^2 + 2x + 3y + 5z = 0$; $2x + 6y - 5z - 6 = 0$ *and passing through the centre of the sphere* $S = x^2 + y^2 + z^2 - 2x - 4y + 6z + 1 = 0$.

Solution. The centre of the sphere $S = 0$ is $(1, 2, -3)$.

The equation of the required sphere is of the form

$x^2 + y^2 + z^2 + 2x + 3y + 5z + \lambda(2x + 6y - 5z - 6) = 0 \qquad \cdots (1)$

It passes through $(1, 2, -3)$.

Hence 1+4+9+2+6-15+λ(2+13-6)=0

$\therefore -7\lambda + 7 = 0$. Hence λ=1

Therefore, from(1), the equation of the sphere is

$$x^2 + y^2 + z^2 + 4x + 9y + 10z - 6 = 0.$$

Problem 2.7.19. *Find the equation of the sphere through the circle* $S \equiv x^2 + y^2 + z^2 - 4 = 0$ *and* $S_1 \equiv x^2 + y^2 + z^2 + 4x - 2y + 4z - 10 = 0$ *and through the point* $(2, 1, 1)$.

Solution. The plane π determined by $S = 0$ and $S_1 = 0$ is

$S - S_1 \equiv -4x + 2y - 4z + 6 = 0$

$\therefore \pi = 2x - y + 2z - 3 = 0$

Now the equation of the required sphere is

$S + \lambda\pi = x^2 + y^2 + z^2 - 4 + \lambda(2x - y + 2z - 3) = 0 \qquad \cdots (1)$

It passes through the point $(2, 1, 1)$.

Hence 4+1+1-4+λ(4-1+2-3)=0.

$\therefore 2\lambda = -2$. Hence $\lambda = -1$

Therefore from(1) the equation of the sphere is

$$x^2 + y^2 + z^2 - 2x + y - 2z - 1 = 0.$$

Problem 2.7.20. *The circle on the sphere* $x^2 + y^2 + z^2 + 6y - 10z + 23 = 0$ *has centre* $(1, 2, -2)$. *Find its equation.*

Solution. The centre of the sphere is $C(0, -3, 5)$. The plane section of the sphere is a circle whose centre is $N(1, 2, -2)$.

Therefore NC is the normal to the intersecting plane. Hence d.r of NC are 1,5,-7.

Therefore the equation of the intersecting plane takes the form x+5y-7z+d=0.

It passes through $(1, 2, -2)$.

Therefore 1+10+14+d=0. Hence $d = -25$.

Hence its equation is $x + 5y - 7z - 25 = 0$.

Therefore the equation of the circle is given by

$x^2 + y^2 + z^2 + 6y - 10z + 23 = 0 = x + 5y - 7z - 25$.

Problem 2.7.21. *Find the centre and radius of the circle determined by the spheres* $S = x^2 + y^2 + z^2 + 10y - 4z - 8 = 0$.

Solution. The centre of the sphere is $C(0, -5, 2)$ and radius

R=$\sqrt{0^2 + 5^2 + (-2)^2 + 8} = \sqrt{37}$

Let O be the centre of the circle of the determined by S=0 and $\pi = 0$.

Therefore CP is perpendicular to the plane $x + y + z - 3 = 0$.

Therefore the direction ratios of CO are $(1, 1, 2)$.

Hence the equation of CO are $\frac{x}{1} = \frac{y+5}{1} = \frac{z-2}{1}$

Any point CO is $(r, r - 5, r + 2)$.

If this point lies on the plane $x + y + z - 3 = 0$, we have $r + (r - 5) + (r + 2) - 3 = 0$.

Therefore $3r = 6$. Hence $r = 2$. Hence O is $(2, -3, 4)$.

Now $CO^2 = (0 - 2)^2 + (-5 + 3)^2 + (2 - 4)^2 = 12$

Radius of the circle $= \sqrt{R^2 - CO^2} = \sqrt{37 - 12} = \sqrt{25} = 5$.

Problem 2.7.22. *If* r *is the radius of the circle given by*

S:$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0; \pi : lx + my + nz = 0$, *prove that*

$(r^2 + d)(l^2 + m^2 + n^2) = (mw - nv)^2 + (nu - lw)^2 + (lv - mu)^2$

Solution. The centre of the sphere $S = 0$ is $C(-u, -v, -w)$ and radius

$R = \sqrt{(u^2 + v^2 + w^2 - d)}$

Let A be the centre of the circle determined by $S = 0$ and $\pi = 0$.

Then CA is perpendicular to the plane $\pi = 0$.

The d.r of CA are the d.r of the normal to the plane $\pi = 0$ and they are l, m, n.

Therefore the equation of the line CA are $\frac{(x+u)}{l} = \frac{(y+v)}{m} = \frac{(z+w)}{n}$.

The point A is $(kl - u, km - v, kn - w)$ for some k.

Since A lies on the plane π, $l((kl - u) + m(km - v) + n(kn - w) = 0$.

$\therefore k(l^2 + m^2 + n^2) = lu + mv + nw.$

Therefore $k = \frac{lu+mv+nw}{l^2+m^2+n^2}$

Now,

$$\begin{aligned}
r^2 &= R^2 - AC^2 \\
&= (u^2 + v^2 + w^2 - d) - [(kl)^2 + (km)^2 + (kn)2] \\
&= (u^2 + v^2 + w^2 - d) - k^2(l^2 + m^2 + n^2) \\
&= (u^2 + v^2 + w^2 - d) - \frac{(lu + mv + nw)^2}{l^2 + m^2 + n^2}(l^2 + m^2 + n^2) \\
\therefore r^2(l^2 + m^2 + n^2) &= (u^2 + v^2 + w^2 - d)(l^2 + m^2 + n^2) - (lu + mv + nw)^2 \\
\therefore (r^2 + d^2)(l^2 + m^2 + n^2) &= (l^2 + m^2 + n^2)(u^2 + v^2 + w^2) - (lu + mv + nw)^2 \\
&= (mw - nv)^2 + (nu - lw)^2 + (lv - mu)^2
\end{aligned}$$

Problem 2.7.23. *Find the equations of the spheres which pass through the circle* $x^2 + y^2 + z^2 - 2x + 2y + 4z - 3 = 0; 2x + y + z - 4 = 0$ *and touch the plane* $3x + 4y - 14 = 0$.

Solution. Let S$=x^2 + y^2 + z^2 - 2x + 2y + 4z - 3 = 0;$

$\pi = 2x + y + z - 4 = 0$

Then $S + \lambda\pi = 0$ represents a sphere passing through the circle determined by S=0 and $\pi = 0$.

$\therefore S = \lambda\pi = x^2 + y^2 + z^2 - 2x + 2y + 4z - 3 + \lambda(2x + y + z - 4) = 0.$

That is, $x^2 + y^2 + z^2 - 2x(1 - \lambda) + y(2 + \lambda) + z(4 + \lambda) - (3 + 4\lambda) = 0 \qquad \cdots (1)$

Center is $\left((1 - \lambda), -\frac{2+\lambda}{2}, -\frac{4+\lambda}{2}\right)$ and

radius is

$=\sqrt{(1-\lambda)^2+(\frac{2+\lambda}{2})^2+(\frac{4+\lambda}{2})^2+(3+4\lambda)}$

Since the sphere touches the plane $3x+4y-14=$, the perpendicular distance from the centre of the sphere to this plane is equal to the radius of the sphere.

$\therefore \frac{3(1-\lambda)-2(2+\lambda)-14}{\sqrt{(3^2+4^2)}} = \sqrt{(1-\lambda)^2+(\frac{2+\lambda}{2})^2+(\frac{4+\lambda}{2})^2+(3+4\lambda)}$

That is, $-2(5\lambda+15) = 5\sqrt{4(1-\lambda)^2+(2+\lambda)^2+(4+\lambda)^2+4(3+4\lambda)}$

$\therefore 100(\lambda+3)^2 = 25[(4+4\lambda^2-8\lambda)+(4+\lambda^2+4\lambda)+(16+\lambda^2+8\lambda)+(12+16\lambda)]$

That is, $4(\lambda^2+6\lambda+9) = 6\lambda^2+20\lambda+36$

That is, $2\lambda^2-4\lambda=0$

Hence $\lambda=0$ or $\lambda=2$

Using these values of λ in (1), we get the equations of the spheres is

$x^2+y^2+z^2-2x+y+z-3=0$; $x^2+y^2+z^2+2x+4y+6z-11=0.$

Problem 2.7.24. *Prove that the circles*

$x^2+y^2+z^2-2x+3y+4z-5=0; 5y+6z+1=0$ *and*

$x^2+y^2+z^2-3x-4y+5z-6=0;\ x+2y-7z=0$ *lie on the same sphere and find its equation.*

Solution. The equation of any sphere through the first circle is

$x^2+y^2+z^2-2x+3y+4z-5+\lambda(5y+6z+1)=0$

That is, $x^2+y^2+z^2-2x+y(3+5\lambda)+2z(2+3\lambda)-5+\lambda=0 \quad \cdots(1)$

The equation of any sphere through the second circle is

$x^2+y^2+z^2-3x-4y+5z-6+\lambda'(x+2y-7z)=0$

That is, $x^2+y^2+z^2-x(3-\lambda')-2y(2-\lambda')+z(5-7\lambda')-6=0$

Equations (1) and (2) will represent the same sphere if

$3-\lambda'=2; -2(2-\lambda')=3+5\lambda; 5-7\lambda'=2(2+3\lambda); -6=-5+\lambda.$

That is, $\lambda'=1$ and $\lambda=-1$ (from the first and last equations).

Also we observe that these values of λ and λ' satisfy the other two equations also.

Hence the two circles lie on the same sphere and its equation is

$x^2+y^2+z^2-2x-2y-2z-6=0$

Problem 2.7.25. *A sphere of constant radius r always passes through the origin and meets the coordinate axes in A, B, C. Prove that the locus of the centroid of the triangle ABC is the sphere $9(x^2+y^2+z^2)=4r^2$*

Solution. Let the equation of the sphere $OABC$ be

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad \cdots (1)$

since it passes through the origin $d = 0$.

Hence $u^2 + v^2 + w^2 = r^2$.

The sphere meets the x-axis at A.

To find x coordinate of A, we put $y = z = 0$ in (1) and we get $x^2 + 2ux = 0$.

Hence $x = -2u$.

Hence A is $(-2u, 0, 0)$. Similarly B is $(0, -2v, 0)$ and C is $(0, 0, -2w)$

The centroid of the triangle ABC is $\left(\frac{-2u}{3}, \frac{-2v}{3}, \frac{-2w}{3}\right) = (x_1, y_1, z_1)$ (say)

$\therefore x_1 = -\frac{2u}{3}; y_1 = \frac{-2v}{3}; z_1 = \frac{-2w}{3}$.

Now, $x_1^2 + y_1^2 + z_1^2 = \frac{4}{9}(u^2 + v^2 + w^2) = \frac{4}{9}r^2$

Therefore the locus of (x_1, y_1, z_1) is $9(x^2 + y^2 + z^2) = 4r^2$.

Problem 2.7.26. *A moving plane intersects the coordinate axes in A, B, C. If the plane always passes through a fixes point (a, b, c) prove that the locus of the centre of the sphere $OABC$ is $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2$.*

Solution. Let the sphere $OABC$ be

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad \cdots (1)$

Since it passes through O we get $d = 0$.

The centre of the sphere is $(-u, -v, -w)$.

The sphere intersects the x-axis at A. To find x-coordinate of A we put $y = z = 0$ in (1) and we get $x^2 + 2ux = 0$. Hence $x = -2u$.

Therefore A is $(-2u, 0, 0)$.

Similarly B is $(0, -2v, 0)$ and $C(0, 0, -2w)$.

The equation of the sphere ABC is $\frac{x}{-2u} + \frac{y}{-2v} + \frac{z}{-2w} = 1$.

Since it passes through the fixed point (a, b, c), we have

$\frac{a}{-2u} + \frac{b}{-2v} + \frac{c}{-2w} = 1$

Now, let (x_0, y_0, z_0) be the centre of the sphere $OABC$ whose locus we to find. Hence $(x_0, y_0, z_0) = (-u, -v, -w)$.

$\therefore u = -x_0, v = -y_0$ and $w = -z_0$

Substituting in (2), we get $\frac{a}{2x_0} + \frac{b}{2y_0} + \frac{c}{2z_0} = 1$

Therefore the locus of (x_0, y_0, z_0) is $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2$

Exercises 2.7.27. 1. Find the equation of the sphere whose centre is $(1, 4, 2)$ and radius 3 units.

2. Find the centre and radius of the sphere $2x^2 + 2y^2 + 2z^2 - 2x + 4y + 2z + 3 = 0$

3. Find the equation of the sphere through the circle $x^2 + y^2 + z^2 = 9; 2x + 3y + 4z = 5$ and the point $(1, 2, 3)$

4. Obtain the equation of the circle lying on the sphere $x^2 + y^2 + z^2 - 2x + 4y - 6z + 3 = 0$ and having its centre at $(2, 3, -4)$.

Chapter 3

UNIT III

3.1 VECTOR DIFFERENTIATION

3.1.1 INTRODUCTION

Vector Calculus is an essential part of Mathematics background required for study of Physics and Chemistry. There are two types of quantities which are defined in Physics, one with direction and the other without direction. Some of the scalar quantities are mass, length, time , volume etc. They are designated with some real number with units. Quantities without direction are called **scalars**. The other kind of quantity is **vector**. It has unit with direction. Some of these types of quantities are displacement, velocity, momentum etc.

Scalar: A Physical Quantity which has magnitude only is called as a Scalar.

Ex: Every Real number is a scalar.

Vector: A Physical Quantity which has both magnitude and direction is called as Vector.

Ex: Velocity, Acceleration.

Geometric description of vectors

We are used to describing the location of any point in the plane by choosing two perpendicular 'coordinate axes' (the x and y axes), and specifying the corresponding (x, y)-coordinates of any given point. In the same way, we can describe where points are in three dimensional space by choosing three mutually perpendicular axes, which

we call the x, y, and z-axes. To say where some given point $P$is, we travel from the origin to P, first along the x-axis, then parallel to the y-axis, and finally parallel to the z-axis. The distances we had to go in the x, y, and z directions are the x, y, and z coordinates of our point P.
We assume that the reader is familiar with the basic results in vector algebra. We give a brief summary of these results in the next section. We denote vectors by bold face Roman letters.

3.1.2 VECTOR ALGEBRA

Through out this chapter **i, j, k** stand for unit vectors along the coordinate axes OX, OY, OZ respectively. If $P(x, y, z)$ is any point, its position vector is given by $\vec{OP} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.
The modulus of $\mathbf{r}$ is given by $|\mathbf{r}| = r = \sqrt{x^2 + y^2 + z^2}$.

Definition 3.1.1. *Let* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *be two vectors. The scalar product or dot product of* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *is defined to be* $\boldsymbol{a}.\boldsymbol{b} = ab\ cos\theta$ *where* θ *is the angle between the two vectors when drawn from a common origin.*

Note 3.1.2. (i) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ (i.e) dot product is commutative.

(ii) $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2 = a^2$

(iii) $\mathbf{a} \cdot \mathbf{b} = 0$ if $\mathbf{a}$ and $\mathbf{b}$ are perpendicular vectors.

(iv) $\mathbf{a} \cdot \mathbf{b} = 0 \Rightarrow \mathbf{a} = 0$ or $\mathbf{b} = 0$ or $\mathbf{a}$ and $\mathbf{b}$ are perpendicular vectors.

(v) $\mathbf{a} \cdot (\mathbf{b}+\mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

(vi) $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$

(vii) $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$

(viii) If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$.

Definition 3.1.3. *Let* $\boldsymbol{a},\boldsymbol{b}$ *be two non zero vectors. Then the* ***vector product*** *or* ***cross product*** *of* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *is a vector perpendicular to both* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *with magnitude* $ab\sin\theta$ *where* $0 \leq \theta \leq \pi$ *is the angle between* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *and whose direction is along a unit vector* $\boldsymbol{n}$ *such that* $\boldsymbol{a},\boldsymbol{b},\boldsymbol{n}$ *form a right handled system.*

Thus $\boldsymbol{a}\times\ \boldsymbol{b}=ab\sin\theta\ \boldsymbol{n}$.

Note 3.1.4. (i) $|\mathbf{a} \times \mathbf{b}|$=**area of the parallelogram with a,b** as adjacent sides.

(ii) $\mathbf{a}\times\mathbf{b} = \mathbf{-b}\times\mathbf{a}$ (i.e) cross product is not commutate.

(iii) $\mathbf{a}\times\mathbf{b} = \mathbf{0}$ if **a** and **b** are parallel.

(iv) $\mathbf{a} \times (\mathbf{b}+\mathbf{c}) = \mathbf{a} \times \mathbf{b}+\mathbf{a}\times \mathbf{c}$

(v) $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$

(vi) $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k}=\mathbf{i}$, $\mathbf{k} \times \mathbf{i}=\mathbf{j}$

(viii) If $\mathbf{a}=a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b}=b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

then $\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Definition 3.1.5. *The **scalar triple product** or **box product** of three vectors **a,b,c** is defined to be the scalar $\mathbf{a}.(\mathbf{b}\times\mathbf{c})$. It is sometimes denoted by [**abc**].*

It can be easily verified that $\mathbf{a}.(\mathbf{b}\times\mathbf{c})=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Note 3.1.6. $\mathbf{a}.(\mathbf{b}\times\mathbf{c})$ represents the volume of the parallelopiped formed by the coterminous edges **a,b,c**.

Note 3.1.7. [**abc**] = [**bca**] = [**cab**]

Note 3.1.8. [**abc**] = -[**bac**] =-[**cba**] = -[**acb**]

Note 3.1.9. The vectors **a,b,c** are coplanar if and only if [**abc**] = 0.

Result 3.1.10. *1.*$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$

2.$(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a}$

3.$(\boldsymbol{a} \times \boldsymbol{b}).(\boldsymbol{c} \times \boldsymbol{d})=\begin{vmatrix} \boldsymbol{a}.\boldsymbol{c} & \boldsymbol{a} \cdot \boldsymbol{d} \\ \boldsymbol{b} \cdot \boldsymbol{c} & \boldsymbol{b} \cdot \boldsymbol{d} \end{vmatrix}$

4.$(\boldsymbol{a} \times \boldsymbol{b}) \times (\boldsymbol{c} \times \boldsymbol{d}) = [\boldsymbol{abd}]\boldsymbol{c} - [\boldsymbol{abc}]\boldsymbol{d}$.

3.1.3 DIFFERENTIATION OF VECTORS

Definition 3.1.11. *Let $\boldsymbol{r}=\boldsymbol{r}(t)=x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}$ be a vector valued function of a scalar variable t.*

$\boldsymbol{r}$ *is said to be* ***differentiable*** *if*

$$\lim_{\Delta t \to 0} = \frac{\boldsymbol{r}(t+\Delta t) - \boldsymbol{r}(t)}{\Delta t}$$

exists and in this case we write

$$\frac{\boldsymbol{dr}}{\boldsymbol{dt}} = \lim_{\Delta t \to 0} \frac{\boldsymbol{r}(t+\Delta t) - \boldsymbol{r}(t)}{\Delta t}$$

.

Theorem 3.1.12. *Let* $\boldsymbol{r} = \boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}$ *be a differentiable function. Then* $\frac{d\boldsymbol{r}}{dt} = x^{'}(t)\boldsymbol{i} + y^{'}(t)\boldsymbol{j} + z^{'}(t)\boldsymbol{k}$.

Proof.

$$\frac{d\mathbf{r}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \left[\frac{x(t+\Delta t) - x(t)}{\Delta t}\right] \mathbf{i} + \left[\frac{y(t+\Delta t) - y(t)}{\Delta t}\right] \mathbf{j} + \left[\frac{z(t+\Delta t) - z(t)}{\Delta t}\right] \mathbf{k}$$

$$= x^{'}(t)\mathbf{i} + y^{'}(t)\mathbf{j} + z^{'}(t)\mathbf{k}$$

□

Theorem 3.1.13. *If* $\boldsymbol{u}(t)=x(t)\boldsymbol{i}+y(t)\boldsymbol{j}+z(t)\boldsymbol{k}$ *and* $\boldsymbol{v}(t)=X(t)\boldsymbol{i}+Y(t)\boldsymbol{j}+Z(t)\boldsymbol{k}$ *then* $\frac{d}{dt}(\boldsymbol{u}.\boldsymbol{v}) = \boldsymbol{u}.\frac{d\boldsymbol{v}}{dt} + \frac{d\boldsymbol{u}}{dt}.\boldsymbol{v}$.

Proof. $\mathbf{u}.\mathbf{v}$=x(t)X(t)+y(t)Y(t)+z(t)Z(t).

$$\begin{aligned}
\therefore \ \frac{d}{dt}(\mathbf{u}.\mathbf{v}) &= x(t)X'(t) + x'(t)X(t) + y(t)Y'(t) + y'(t)Y(t) + z(t)Z'(t) + z'(t)Z(t). \\
&= [x(t)X'(t) + y(t)Y'(t) + z(t)Z'(t)] + \\
&\qquad [x'(t)X(t) + y'(t)Y(t) + z'(t)Z(t)] \\
&= \mathbf{u}.\frac{d\mathbf{v}}{dt} + \frac{d\mathbf{u}}{dt}.\mathbf{v}
\end{aligned}$$

□

Theorem 3.1.14. $\frac{d}{dt}(\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{u} \times \frac{d\boldsymbol{v}}{dt} + \frac{d\boldsymbol{u}}{dt} \times \boldsymbol{v}$

Proof. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x(t) & y(t) & z(t) \\ X(t) & Y(t) & Z(t) \end{vmatrix}$

$= [y(t)Z(t) - Y(t)z(t)]\mathbf{i} - [x(t)Z(t) - X(t)z(t)]\mathbf{j} + x(t)Y(t) - X(t)y(t)]\mathbf{k}$

$$\begin{aligned}
\therefore \quad \frac{d}{dt}(\mathbf{u} \times \mathbf{v}) &= [(y(t)Z'(t) + y'(t)Z(t) - (Y(t)z'(t) + Y'(t)z(t))]\mathbf{i} \\
&\qquad - [(x(t)Z'(t) + x'(t)Z(t) - (X(t)z'(t) + X'(t)z(t))]\mathbf{j} \\
&\qquad + [(x(t)Y'(t) + x'(t)Y(t) - (X(t)y'(t) + X'(t)y(t))]\mathbf{k} \\
&= [(y(t)Z'(t) - y'(t)Z(t))]i - [(x(t)Z'(t) - X'(t)z(t))]j \\
&\qquad + [(x(t)Y'(t) - X'(t)y(t))]k + [(y'(t)Z(t) - y(t)Z'(t))]i \\
&\qquad - [(x'(t)Z(t) - X'(t)z(t))]j + [(x'(t)Y(t) - X'(t)y(t))]k \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x(t) & y(t) & z(t) \\ X'(t) & Y'(t) & Z'(t) \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x'(t) & y'(t) & z'(t) \\ X(t) & Y(t) & Z(t) \end{vmatrix} \\
&= \mathbf{u} \times \frac{d\mathbf{v}}{dt} + \frac{d\mathbf{u}}{dt} \times \mathbf{v}
\end{aligned}$$

□

Theorem 3.1.15. $\frac{d(f\boldsymbol{u})}{dt} = f\frac{d\boldsymbol{u}}{dt} + \frac{df}{dt}\boldsymbol{u}$ *where f is a scalar values function $f(t)$.*

Proof is left as an exercise.

Theorem 3.1.16. $\frac{d}{dt}[\boldsymbol{fgh}] = \left[\boldsymbol{fg}\ \frac{d\boldsymbol{h}}{dt}\right] + \left[\boldsymbol{f}\ \frac{d\boldsymbol{g}}{dt}\boldsymbol{h}\right] + \left[\frac{d\boldsymbol{f}}{dt}\ \boldsymbol{gh}\right]$

Proof.

$$\begin{aligned}
\frac{d}{dt}[\mathbf{fgh}] &= \frac{d}{dt}\{\mathbf{f}.(\mathbf{g} \times \mathbf{h})\} = \frac{d\mathbf{f}}{dt}.(\mathbf{g} \times \mathbf{h}) + \mathbf{f}.\frac{d}{dt}(\mathbf{g} \times \mathbf{h}) \\
&= \frac{d\mathbf{f}}{dt}.(\mathbf{g} \times \mathbf{h}) + \mathbf{f}.\left(\mathbf{g} \times \frac{d\mathbf{h}}{dt} + \frac{d\mathbf{g}}{dt} \times \mathbf{h}\right) \\
&= \left[\frac{d\mathbf{f}}{dt}\mathbf{gh}\right] + \left[\mathbf{f}\frac{d\mathbf{g}}{dt}\mathbf{h}\right] + \left[\mathbf{fg}\frac{d\mathbf{h}}{dt}\right].
\end{aligned}$$

□

3.1.4 Solved problems

Problem 3.1.17. *If* $\boldsymbol{r}= \boldsymbol{a}\cos\omega t + \boldsymbol{b}\sin\omega t$ *where* $\boldsymbol{a},\boldsymbol{b}$ *are constant vectors and* ω *is a constant, prove that* $\boldsymbol{r}\times\frac{d\boldsymbol{r}}{dt} = \omega(\boldsymbol{a}\times\boldsymbol{b})$ *and* $\frac{d^2\boldsymbol{r}}{dt^2}+\omega^2\boldsymbol{r} = 0$.

Solution.

$$\begin{aligned}\frac{d\mathbf{r}}{dt} &= -\mathbf{a}\omega\sin\omega t+\mathbf{b}\omega\cos\omega t\\ \frac{d^2\mathbf{r}}{dt^2} &= -\mathbf{a}\omega^2\cos\omega t-\mathbf{b}\omega^2\sin\omega t\\ \frac{d^2\mathbf{r}}{dt^2} &= -\omega^2(\mathbf{a}\cos\omega t+\mathbf{b}\sin\omega t)\end{aligned}$$

$\therefore \frac{d^2\mathbf{r}}{dt^2}+\omega^2\mathbf{r}=0$

$$\begin{aligned}\text{Now, }\ \mathbf{r}\times\frac{d\mathbf{r}}{dt} &= (\mathbf{a}\cos\omega t+\mathbf{b}\sin\omega t)\times(-\mathbf{a}\omega\sin\omega t+\mathbf{b}\omega\cos\omega t)\\ &= \omega\mathbf{a}\times\mathbf{b}\cos\omega^2 t-\omega\mathbf{b}\times\mathbf{a}\sin^2\omega t\\ &= \omega\mathbf{a}\times\mathbf{b}\cos\omega^2 t+\omega\mathbf{a}\times\mathbf{b}\sin^2\omega t\\ &= \omega(\mathbf{a}\times\mathbf{b}).\end{aligned}$$

Problem 3.1.18. *If* $\boldsymbol{u}(t)$ *is a vector which is constant in magnitude prove that* $\frac{d\boldsymbol{u}}{dt}=0$ *or* $\frac{d\boldsymbol{u}}{dt}$ *is perpendicular to* $\boldsymbol{u}$.

Solution. $\mathbf{u}.\mathbf{u}=c$ (a constant) $\Rightarrow \frac{d\mathbf{u}}{dt}.\mathbf{u}+\mathbf{u}.\frac{d\mathbf{u}}{dt}=0$. Hence $\mathbf{u}.\frac{d\mathbf{u}}{dt}=0$
$\therefore \frac{d\mathbf{u}}{dt}=0$ or $\frac{d\mathbf{u}}{dt}$ is perpendicular to $\mathbf{u}$.

Exercises 3.1.19.

1. If $\mathbf{r}=\mathbf{a}e^{\omega t}+\mathbf{b}e^{-\omega t}$ show that $\frac{d^2\mathbf{r}}{dt^2}-\omega^2\mathbf{r}=0$ where a and b are constant vectors.
2. Differentiate $\left(\mathbf{r}.\frac{d\mathbf{r}}{dt}\right)$ with respect to t.
3. Expand $\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right)=\frac{1}{r}\frac{d\mathbf{r}}{dt}-\frac{1}{r^2}\frac{dr}{dt}\mathbf{r}$.

3.2 GRADIENT

In differential calculus, we have introduced the operator $\frac{d}{dx}$. When applied to a differentiable function $f(x)$ it yields another functioin $\frac{df}{dx}$. In this section we introduce another operator∇(to be read as **del**) given by

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}.$$

Definition 3.2.1. *Let $\varphi(x, y, z)$ be a real valued function having continuous first order partial derivatives. We define* $\nabla\varphi = \boldsymbol{i}\frac{\partial\varphi}{\partial x} + \boldsymbol{j}\frac{\partial\varphi}{\partial y} + \boldsymbol{k}\frac{\partial\varphi}{\partial z} = \sum \boldsymbol{i}\frac{\partial\varphi}{\partial x}$.

$\nabla\varphi$ is called **gradient** of φ and is denoted by *grad* φ. Thus, the gradient of a scalar function φ is a vector valued function.

Example 3.2.2. If $\varphi(x, y, z) = xy^2 + yz^3$ then

$$\begin{aligned} grad\ \varphi = \nabla\varphi &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)(xy^2 + yz^3) \\ &= y^2\mathbf{i} + (2xy + z^3)\mathbf{j} + 3yz^2\mathbf{k} \end{aligned}$$

3.2.1 Geometrical interpretation

Let $\varphi(x, y, z)$ be a scalar valued function having continuous partial derivatives. Let $P(x_0, y_0, z_0)$ be any point. Let $\varphi(x_0, y_0, z_0) = c$.

Then the equation $\varphi(x, y, z) = c$ represents a **surface**. Obviously (x_0, y_0, z_0) lies on this surface. Along this surface $d\varphi = 0$.

That is, $\frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz = 0$

That is, $\nabla\varphi.d\mathbf{r} = 0$ where $d\mathbf{r} = \mathbf{i}dx + \mathbf{j}dy + \mathbf{k}dz$.

$\therefore$ $\nabla\varphi$ is perpendicular to $d\mathbf{r}$ as long as $d\mathbf{r}$ represents a change from P to Q where Q remains on the surface $\varphi(x, y, z) = c$

$\therefore$ $\nabla\varphi$ is normal to all the tangents to the surface at $P(x_0, y_0, z_0)$.

Hence $\nabla\varphi$ **represents the normal to the surface** $\varphi(x, y, z) = c$.

Hence the unit normal to $\mathbf{n}$ to the surface $\varphi(x, y, z) = c$ is given by $\mathbf{n}$=$\frac{\nabla\varphi}{|\nabla\varphi|}$

Definition 3.2.3. *Let* $\boldsymbol{a}$ *be a unit vector. The component of the vector* $\nabla\varphi$ *in the direction* $\boldsymbol{a}$ *is given by* $\boldsymbol{a}.\nabla\varphi$ *and is called the* ***directional derivative*** *of* φ *in the direction* $\boldsymbol{a}$*. This can be interpretted as the rate of change of* φ *at* (x, y, z) *in the direction* $\boldsymbol{a}$

Note 3.2.4. Let $P = (x, y, z)$ and $Q = (x + \Delta x, y + \Delta y, z + \Delta z)$ be two neighbouring points and Δs be the distance between P and Q.

$$\text{Then } \frac{d\varphi}{ds} = \frac{\partial\varphi}{\partial x}\frac{dx}{ds} + \frac{\partial\varphi}{\partial y}\frac{dy}{ds} + \frac{\partial\varphi}{\partial z}\frac{dz}{ds} = \frac{d\mathbf{r}}{ds}.\nabla\varphi$$

Since $\frac{dr}{ds}$ is a unit vector $\frac{dr}{ds}.\nabla\varphi$ is the directional derivative of φ in the direction of $\frac{dr}{ds}$.
$\therefore \frac{d\varphi}{ds} = \frac{dr}{ds}.\nabla\varphi$ has a maximum value when $\nabla\varphi$ and $\frac{dr}{ds}$ have the same directions.
Therefore *the maximum value of the directional derivative takes place in the direction of* $\nabla\varphi$ *and its magnitude is* $|\nabla\varphi|$.

Equation of the tangent plane to the surface to $\varphi(x, y, z) = c$ at a point $A(x_0, y_0, z_0)$.

Let $P(x, y, z)$ be any point on the tangent plane whose position vector is $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.
A is the point of contact of the tangent plane with the surface whose position vector is $\mathbf{r}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$.
Then $\mathbf{r}$-$\mathbf{r}_0$ is a vector on the tangent plane. $(\nabla\varphi)$ at (x_0, y_0, z_0) is the normal to the surface and hence perpendicular to the tangent plane.
At the point (x_0, y_0, z_0), $(\mathbf{r}\text{-}\mathbf{r}_0).(\nabla\varphi) = 0$ $\cdots(1)$
Since it is true for all points $\mathbf{r}$ on the tangent plane, (1) represents the equation of the tangent plane.

Equation of the normal line

Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be any point on the normal line at $A(x_0, y_0, z_0)$ whose position vector is $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Hence $\mathbf{r}$-$\mathbf{r}_0$ lies along the normal line at A. Hence $\nabla\varphi$ at (x_0, y_0, z_0) is parallel to $r - r_0$ so that at (x_0, y_0, z_0), $(r - r_0) \times \nabla\varphi = 0$ $\cdots(2)$

Since it is true for all points $\mathbf{r}$ on the normal line (2) represents the equation of the normal.

Equation of the (i) **tangent line** (ii) **normal plane** at a given point

$A(x_0, y_0, z_0)$ of the curve which is the intersection of the two surfaces $\varphi(x, y, z) = c_1$ and $\psi(x, y, z) = c_2$.

(i) Let C be the curve along which the two surfaces intersect.

Let $A(x_0, y_0, z_0)$ be a point on C whose position vector is $r_0 = x_0 i + y_0 j + z_0 k$. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be any point on the tangent line at A to the curve C.

$\nabla\varphi$ at (x_0, y_0, z_0) and $\nabla\psi$ at (x_0, y_0, z_0) represent the normals to the surface $\varphi = C_1$ and $\psi = C_2$ respectively and both these are perpendicular to the tangent line at A.

Therefore $\mathbf{r}$-$\mathbf{r}_0$ is parallel to $(\nabla\varphi \times \nabla\psi)$ at (x_0, y_0, z_0) so that (x_0, y_0, z_0), $\mathbf{r}$-$\mathbf{r}_0 \times (\nabla\varphi \times \nabla\psi)$ represents the equation of the tangent line at A.

(ii) Also the equation of the normal plane at (x_0, y_0, z_0) is given by at (x_0, y_0, z_0), $(\mathbf{r}\text{-}\mathbf{r}_0) \cdot (\nabla\varphi \times \nabla\psi) = 0$.

Theorem 3.2.5. $\boldsymbol{grad}(\varphi \pm \psi) = grad\varphi \pm grad\psi$

Proof.

$$\begin{aligned}
\mathbf{grad}(\varphi \pm \psi) &= \nabla(\varphi \pm \psi) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)(\varphi \pm \psi) \\
&= \left(\mathbf{i}\frac{\partial\varphi}{\partial x} + \mathbf{j}\frac{\partial\varphi}{\partial y} + \mathbf{k}\frac{\partial\varphi}{\partial z}\right) \pm \left(\mathbf{i}\frac{\partial\psi}{\partial x} + \mathbf{j}\frac{\partial\psi}{\partial y} + \mathbf{k}\frac{\partial\psi}{\partial z}\right) \\
&= \nabla\varphi \pm \nabla\psi. \\
&= \mathbf{grad}\varphi \pm \mathbf{grad}\psi
\end{aligned}$$

□

Theorem 3.2.6. $grad(\varphi\psi) = \varphi grad\psi + \psi grad\varphi$

Proof.

$$\begin{aligned}
grad(\varphi\psi) &= \sum i\left(\frac{\partial}{\partial x}(\varphi\psi)\right) = \sum i\left(\varphi\frac{\partial\psi}{\partial x} + \psi\frac{\partial\varphi}{\partial x}\right) \\
&= \sum i\varphi\frac{\partial\psi}{\partial x} + \sum \mathbf{i}\psi\frac{\partial\varphi}{\partial x} = \varphi\left(\sum \mathbf{i}\frac{\partial\psi}{\partial x}\right) + \psi\left(\sum \mathbf{i}\frac{\partial\varphi}{\partial x}\right) \\
&= \varphi(\nabla\psi) + \psi(\nabla\varphi) = \varphi\ grad\psi + \psi\ grad\varphi
\end{aligned}$$

□

Theorem 3.2.7. $grad\ \left(\frac{\varphi}{\psi}\right) = (\psi grad\ \varphi - \varphi grad\ \psi)/\psi^2$

Proof.

$$\begin{aligned} grad\left(\frac{\varphi}{\psi}\right) &= \nabla\left(\frac{\varphi}{\psi}\right) = \sum \mathbf{i}\frac{\partial}{\partial x}\left(\frac{\varphi}{\psi}\right) = \sum \mathbf{i}\left[\frac{\psi\frac{\partial\varphi}{\partial x} - \varphi\frac{\partial\psi}{\partial x}}{\psi^2}\right] \\ &= \frac{1}{\psi^2}\left[\sum \mathbf{i}\psi\frac{\partial\varphi}{\partial x} - \sum \mathbf{i}\varphi\frac{\partial\psi}{\partial x}\right] \\ &= (\psi grad\varphi - \varphi grad\psi)/\psi^2 \end{aligned}$$

□

3.2.2 Solved problems

Problem 3.2.8. *If* $\boldsymbol{r}$ *is the position vector of any point* $P(x, y, z)$*, prove that* $grad\ r^n = nr^{n-2}r$*.*

Solution. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Then $r^2 = x^2 + y^2 + z^2$.
$\therefore 2\mathbf{r}\left(\frac{\partial \mathbf{r}}{\partial x}\right) = 2x$. Hence $\frac{\partial r}{\partial x} - \frac{x}{\mathbf{r}}$. Similarly, $\frac{\partial r}{\partial y} = \frac{y}{\mathbf{r}}$ and $\frac{\partial \mathbf{r}}{\partial z} = \frac{z}{\mathbf{r}}$

$$\begin{aligned} \text{Now, gradr}^n &= \nabla\mathbf{r}^n = \mathbf{i}\frac{\partial\mathbf{r}^n}{\partial x} + \mathbf{j}\frac{\partial\mathbf{r}^n}{\partial y} + \mathbf{k}\frac{\partial\mathbf{r}^n}{\partial z} \\ &= \mathbf{i}n\mathbf{r}^{n-1}\frac{\partial\mathbf{r}}{\partial x} + \mathbf{j}n\mathbf{r}^{n-1}\frac{\partial\mathbf{r}}{\partial y} + \mathbf{k}n\mathbf{r}^{n-1}\frac{\partial\mathbf{r}}{\partial z} \\ &= n\mathbf{r}^{n-1}\left[\mathbf{i}\frac{x}{\mathrm{r}} + \mathbf{j}\frac{y}{\mathrm{r}} + \mathbf{k}\frac{z}{\mathrm{r}}\right] = n\mathbf{r}^{n-2}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \\ &= n\mathrm{r}^{n-2}\mathbf{r} \end{aligned}$$

Problem 3.2.9. *If* $\varphi(x, y) = log\sqrt{x^2 + y^2}$ *show that*

$$grad\ \varphi = \frac{\boldsymbol{r} - (\boldsymbol{k}.\boldsymbol{r})\boldsymbol{k}}{\{\boldsymbol{r}\text{-}(\boldsymbol{k}.\boldsymbol{r})\boldsymbol{k}\}.\{\boldsymbol{r}\text{-}(\boldsymbol{k}.\boldsymbol{r})\boldsymbol{k}\}}$$

Solution.

$$\begin{aligned}
grad\ \varphi &= \nabla log\sqrt{x^2+y^2} = \frac{1}{2}\nabla log(x^2+y^2) \\
&= \frac{1}{2}\left(\mathbf{i}\frac{\partial\varphi}{\partial x}+\mathbf{j}\frac{\partial\varphi}{\partial y}+\mathbf{k}\frac{\partial\varphi}{\partial z}\right) log(x^2+y^2) \\
&= \frac{1}{2}\left[\mathbf{i}\left(\frac{2x}{x^2+y^2}\right)+\mathbf{j}\left(\frac{2y}{x^2+y^2}\right)+\mathbf{k}(0)\right] \\
&= \frac{x\mathbf{i}+y\mathbf{j}}{(x\mathbf{i}+y\mathbf{j}).(x\mathbf{i}+y\mathbf{j})} \\
&= \frac{\mathbf{r}-z\mathbf{k}}{(\mathbf{r}-z\mathbf{k}).(\mathbf{r}-z\mathbf{k})} \quad \text{(since } \mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}) \\
&= \frac{\textbf{r-(k.r)k}}{\textbf{\{r-(k.r)k\}.\{r-(k.r)k\}}}\text{(since } \mathbf{k.r}=z)
\end{aligned}$$

Problem 3.2.10. *Show that* φ***(a.r)=a*** *for any constant vector* ***a****.*

Solution. Let $a = a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$

Therefore $\mathbf{a.r} = a_1x+a_2y+a_3z.$

$$\nabla(\mathbf{a.r}) = \left(\mathbf{i}\frac{\partial\varphi}{\partial x}+\mathbf{j}\frac{\partial\varphi}{\partial y}+\mathbf{k}\frac{\partial\varphi}{\partial z}\right)(a_1x+a_2y+a_3z) = a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}=\mathbf{a}$$

.

Problem 3.2.11. *Obtain the directional derivative of* $\varphi = xy^2+yz^3$ *at the point (2,-1,1) in the direction of* $\boldsymbol{i}+2\boldsymbol{j}+2\boldsymbol{k}$*.*

Solution. $\nabla\varphi = y^2\mathbf{i}+(2xy+z^3)\mathbf{j}+3yz^2\mathbf{k}$

At (2,-1,1), we get $\nabla\varphi = \mathbf{i}-3\mathbf{j}-3\mathbf{k}$.

The unit vector of the given direction a is ($\mathbf{i}$+2$\mathbf{j}$+2$\mathbf{k}$)/3.

Therefore the required directional derivative is $\mathbf{a}.\nabla\varphi = -11/3$

Problem 3.2.12. *Find the unit normal to the surface* $x^3-syz+z^3=1$ *at (1,1,1).*

Solution. Let $\varphi = x^3-syz+z^3-1$.

Let $\mathbf{n}$ denote the unit normal to the surface.

Then $\mathbf{n}=\frac{\nabla\varphi}{|\nabla\varphi|}$

Now, $\nabla\varphi = (3x^2 - yz)\mathbf{i} - xz\mathbf{j} + (3z^2 - xy)\mathbf{k}$.

$\therefore$ $\nabla\varphi$ at (1,1,1) $= 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. Hence $\mathbf{n}=\frac{1}{3}(2\mathbf{i} - \mathbf{j} + 2\mathbf{k})$

Problem 3.2.13. *If $\nabla\varphi = 2xyz^3\boldsymbol{i} + x^2z^3\boldsymbol{j} + 3x^2yz^2\boldsymbol{k}$ then find $\Phi(x, y, z)$ if $\Phi(1, -2, 2) = 4$*

Solution. $\nabla\varphi = \left(\mathbf{i}\frac{\partial\varphi}{\partial x} + \mathbf{j}\frac{\partial\varphi}{\partial y} + \mathbf{k}\frac{\partial\varphi}{\partial z}\right) = 2xyz^3\mathbf{i} + x^2z^3\mathbf{j} + 3x^2yz^2\mathbf{k}$.

$\therefore \frac{\partial\Phi}{\partial x} = 2xyz^3 \quad \cdots(1)$

$\frac{\partial\Phi}{\partial y} = x^2z^3 \quad \cdots(2)$

$\frac{\partial\Phi}{\partial z} = 3x^2yz^2 \quad \cdots(3)$

Integrating (1),(2),(3) w.r.to x, y, z respectively we get,

$$\Phi = yz^3x^2 + f(y, z); \Phi = x^2yz^3 + g(x, z); \Phi = x^2yz^3 + h(x, y)$$

$\therefore \Phi = x^2yz^3 + k$ where k is a constant.

Given $\varphi(1, -2, 2) = 4$. Hence $4 = -16 + k$. Hence $k = 20$.

$\therefore \Phi(x, y, z) = x^2yz^3 + 20$

Problem 3.2.14. *Find the equation of the (i) tangent plane and (ii) normal line to the surface $xyz = 4$ at the point (1,2,2).*

Solution. Let $\varphi = xyz - 4$

$\nabla\varphi = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

At (1,2,2) $\nabla\varphi = 4\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.

The position vector of (1,2,2) is $r_0 = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.

(i) The equation of the tangent plane is given by $(\mathbf{r}\text{-}\mathbf{r}_0).\nabla\varphi = 0$

$$\therefore [(x - 1)\mathbf{i} + (y - 2)\mathbf{j} + (z - 2)\mathbf{k}].(4\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) = 0$$

$$\therefore 4(x - 1) + 2(y - 2) + 2(z - 2) = 0$$

$\therefore 2x + y + z = 6$

(ii) The equation of the normal line at $(1, 2, 2)$ is given by $(\mathbf{r}\text{-}\mathbf{r}_0) \times \nabla\varphi = 0$

$$\therefore \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x-1 & y-2 & z-2 \\ 4 & 2 & 2 \end{vmatrix} = \mathbf{0}$$

$[2(y-2) - 2(z-2)]\mathbf{i} - [2(x-1) - 4(z-2)]\mathbf{j} + [2(x-1) - 4(y-2)]\mathbf{k} = 0$

Equating the coefficients of **i,j,k** on both sides,

$(y-2) = (z-2); (x-1) = 2(z-2); (x-1) = 2(y-2)$ which can be written in symmetric form in rectangular cartesian coordinates as $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-2}{1}$.

Problem 3.2.15. *Find the angle between the surfaces* $x^2 + y^2 + z^2 = 29$ *and* $x^2 + y^2 + z^2 + 4x - 6y - 8z - 47 = 0$ *at (4,-3,2).*

Solution. Let $\varphi(x, y, z) = x^2 + y^2 + z^2 - 20 = 0$ $\cdots(1)$

$\psi(x, y, z) = x^2 + y^2 + z^2 + 4x - 6y - 8z - 47$ $\cdots(2)$

$\nabla\varphi = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}; \nabla\varphi$ at (4,-3,2)=8**i**-6**j**+4**k** $\cdots(3)$

$\nabla\psi = (2x + 4)\mathbf{i} + (2y - 6)\mathbf{j} + (2z - 8)\mathbf{k}; \nabla\psi$ at (4.-3,2)=12**i**-12**j**-4**k** $\cdots(4)$

We know that the angle between two surfaces is the angle between the tangent planes at a common point and hence the angle between the normals at that point.

Equations (3) and (4) represent the normal to the surfaces (1) and (2) at (4,-3,2) respectively

Let θ be the angle between the normals (3) and (4) at (4,-3,2)

$$\begin{aligned}
\therefore \cos\theta &= \frac{\nabla\varphi.\nabla\psi}{|\nabla\varphi||\nabla\psi|} = \frac{96 + 72 - 16}{\sqrt{8^2 + (-6)^2 + 4^2}\sqrt{12^2 + (-12)^2 + (-4)^2}} \\
&= \frac{152}{\sqrt{116}\sqrt{304}} = \frac{19}{\sqrt{29}\sqrt{19}} = \sqrt{19/29} \\
\therefore \quad \theta &= \cos^{-1}\sqrt{19/29}
\end{aligned}$$

Problem 3.2.16. *Determine the constants* a *and* b *so that the surface* $5x^2 - 2yz - 9x = 0$ *will be orthogonal to the surface* $ax^2y + bz^2 = 4$ *at the point* $(1, -1, 2)$.

Solution. Let $\varphi(x,y,z) = 5x^2 - 2yz - 9x$ and $\psi(x,y,z) = ax^2y + bz^2 - 4$

$\therefore\ \nabla\varphi = (10x-9)\mathbf{i} - 2z\mathbf{j} - 2y\mathbf{k}$ and $\nabla\psi = 2ax\mathbf{i} + ax^2\mathbf{j} + 2bz\mathbf{k}$.

Therefore at (1,-1,2), $\nabla\varphi = \mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$ and $\nabla\psi = -2a\mathbf{i} + a\mathbf{j} + 4b\mathbf{k}$.

The two surfaces will be orthogonal at (1,-1,2) if the surface normals to the two surfaces at (1,-1,2) are perpendicular.

Hence $\nabla\varphi.\nabla\psi = 0$ at $(1,-1,2)$.

$\therefore\ (\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}).(-2a\mathbf{i} + a\mathbf{j} + 4b\mathbf{k}) = 0$

Therefore $-2a - 4b + 8b = 0$. That is, $8b - 6a = 0$ $\cdots(1)$

Further $(1,-1,2)$ lies on both the surfaces,

Taking $\psi(1,-1,2) = 0$, we have $-a + 4b = 4$ $\cdots(2)$

Solving (1) and (2), we get $a = 2$ and $b = 3/2$.

Exercises 3.2.17. 1. Find the grad Φ for the following at the points indicated.

(i) $\Phi(x,y,z) = zx - y^2$ at (a,b,c).

(ii) $\Phi(x,y,z) = xyz$ at (x_1, y_1, z_1).

2. If $\Phi = x^2z + e^{y/x}$ and $\psi = 2z^2y - xy^2$ find $\nabla(\Phi + \psi)$ and $\nabla(\Phi\psi)$ at $(1,0,2)$.

3. Find the unit normal to the surface $xy^3z^2 = 4$ at $(-1,-1,2)$.

4. Find the equation of the tangent plane at the origin to the surface $x^2 + y^2 + z^2 + 8x - 6y + 4z = 0$.

5. Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2,-1,2)$.

6. Find the directional derivative of $\varphi = xy + yz + zx$ at the point $(1,2,3)$ in the direction of $3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$.

7. Find the directional derivative of $\varphi = 4\ e^{2x-y+z}$ at the point $(1,1,-1)$ in the direction towards the point $(-3,5,6)$.

3.3 DIVERGENCE AND CURL

Definition 3.3.1. *Let* $\boldsymbol{f} = f_1\boldsymbol{i} + f_2\boldsymbol{j} + f_3\boldsymbol{k}$ *be a vector valued function. The* ***divergence*** *of* $\boldsymbol{f}$ *denoted by* $\nabla.\boldsymbol{f}$ *or div* $\boldsymbol{f}$ *is defined by*

$$\nabla.\boldsymbol{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \sum \boldsymbol{i}.\frac{\partial \boldsymbol{f}}{\partial x}$$

The ***curl*** *of* $\boldsymbol{f}$ *denoted by* $\nabla \times \boldsymbol{f}$ *or* ***curl*** $\boldsymbol{f}$ *is defined by*

$$\boldsymbol{curl}\ \boldsymbol{f} = \sum \boldsymbol{i} \times \frac{\partial \boldsymbol{f}}{\partial x} = \boldsymbol{i}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) + \boldsymbol{j}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) - \boldsymbol{k}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

Note 3.3.2. The divergence of a vector valued function is a scalar valued function.

Note 3.3.3. The curl of a vector valued function is a vector valued function.

Note 3.3.4. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ the symbol $\mathbf{a}.\ \nabla$ stands for the operator $a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}$.

Examples 3.3.5. 1) Let$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

$div\mathbf{r} = \nabla.\mathbf{r} = 1 + 1 + 1 = 3$ and $curl\ \mathbf{r} = 0$

2) Let $\mathbf{f} = xz^3\mathbf{i} - 2x^2yz\mathbf{j} + 2yz^4\mathbf{k}$.

Then $\nabla.\mathbf{f} = z^3 - 2x^2z + 8yz^3$ and $\nabla \times \mathbf{f} = (2z^4 + 2x^2y)\mathbf{i} + 3xz^2\mathbf{j} - 4xyz\mathbf{k}$ (verify)

Definition 3.3.6. *A vector* $\boldsymbol{f}$ *is called* ***solenoidal*** *if div* $\boldsymbol{f} = 0$.

A vector $\boldsymbol{f}$ *is called* ***irrotational*** *if curl*$\boldsymbol{f} = 0$

Theorem 3.3.7. *div* $(\boldsymbol{f}+\boldsymbol{g}) = div\ \boldsymbol{f} + div\ \boldsymbol{g}$.

Proof. Let $\mathbf{f} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$ and $\mathbf{g} = g_1\mathbf{i} + g_2\mathbf{j} + g_3\mathbf{k}$

$$\begin{aligned} div(\mathbf{f}+\mathbf{g}) = \nabla.\mathbf{f}+\mathbf{g} &= \frac{\partial}{\partial x}(f_1+g_1) + \frac{\partial}{\partial y}(f_2+g_2) + \frac{\partial}{\partial z}(f_3+g_3) \\ &= \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) + \left(\frac{\partial g_1}{\partial x} + \frac{\partial g_2}{\partial y} + \frac{\partial g_3}{\partial z}\right) \\ &= \nabla.\mathbf{f} + \nabla.\mathbf{g} = div\mathbf{f} + div\mathbf{g} \end{aligned}$$

□

Theorem 3.3.8. *Let $\boldsymbol{f}$ be a vector valued function and Φ a scalar valued function. Then $\nabla.(\Phi\boldsymbol{f}) = (\nabla\Phi).\boldsymbol{f} + (\nabla.\boldsymbol{f})\Phi$*
That is, $div(\Phi\boldsymbol{f}) = (grad\phi).\boldsymbol{f} + (div\boldsymbol{f})\phi$.

Proof. Let $\mathbf{f} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$.

$$\begin{aligned} \nabla.(\Phi\mathbf{f}) &= \nabla.(\Phi f_1\mathbf{i} + \Phi f_2\mathbf{j} + \Phi f_3\mathbf{k}) \\ &= \frac{\partial}{\partial x}(\Phi f_1) + \frac{\partial}{\partial y}(\Phi f_2) + \frac{\partial}{\partial z}(\Phi f_3) \\ &= \left(\Phi\frac{\partial f_1}{\partial x} + f_1\frac{\partial \Phi}{\partial x}\right) + \left(\Phi\frac{\partial f_2}{\partial y} + f_2\frac{\partial \Phi}{\partial y}\right) + \left(\Phi\frac{\partial f_3}{\partial z} + f_3\frac{\partial \Phi}{\partial z}\right) \\ &= \left(\frac{\partial \Phi}{\partial x}i + \frac{\partial \Phi}{\partial y}j + \frac{\partial \Phi}{\partial z}k\right).(f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}) + \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right)\Phi \\ &= (\nabla\Phi).\mathbf{f} + (\nabla.\mathbf{f})\Phi. \end{aligned}$$

□

Theorem 3.3.9. $\nabla.(\boldsymbol{f}\times\boldsymbol{g}) = \boldsymbol{g}.(\nabla\times\boldsymbol{f}) - \boldsymbol{f}.(\nabla\times\boldsymbol{g})$
That is, $div(\boldsymbol{f}\times\boldsymbol{g}) = \boldsymbol{g}.curl\boldsymbol{f} - \boldsymbol{f}.curl\boldsymbol{g}$.

Proof.

$$\begin{aligned} div(\mathbf{f}\times\mathbf{g}) &= \nabla.(\mathbf{f}\times\mathbf{g}) = \sum \mathbf{i}.\left(\frac{\partial}{\partial x}(\mathbf{f}\times\mathbf{g})\right) \\ &= \sum \mathbf{i}\left(\frac{\partial \mathbf{f}}{\partial x}\times\mathbf{g} + \mathbf{f}\times\frac{\partial \mathbf{g}}{\partial x}\right) = \sum \mathbf{i}\left(\frac{\partial \mathbf{f}}{\partial x}\times\mathbf{g}\right) + \sum \mathbf{i}\left(\mathbf{f}\times\frac{\partial \mathbf{g}}{\partial x}\right) \\ &= \sum\left(\mathbf{i}\times\frac{\partial \mathbf{f}}{\partial x}\right).\mathbf{g} - \sum\left(\mathbf{i}\times\frac{\partial \mathbf{g}}{\partial x}\right).\mathbf{f} \\ &= (\nabla\times\mathbf{f}).\mathbf{g} - (\nabla\times\mathbf{g}).\mathbf{f} \end{aligned}$$

□

Theorem 3.3.10. *div grad*$\Phi = \nabla.\nabla\Phi = \nabla^2\Phi$ *where* $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

Proof. $\nabla\Phi = \frac{\partial\Phi}{\partial x}\mathbf{i} + \frac{\partial\Phi}{\partial y}\mathbf{j} + \frac{\partial\Phi}{\partial z}\mathbf{k}$

$$\therefore \nabla.\nabla\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = \nabla^2\Phi.$$

□

Note 3.3.11. The operator ∇^2 is called the **Laplacian operator**. If Φ is a scalar valued function, $\nabla^2\Phi$ is also a scalar valued function. If $\mathbf{f} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$, we define $\nabla^2\mathbf{f} = (\nabla^2 f_1)\mathbf{i} + (\nabla^2 f_2)\mathbf{j} + (\nabla^2 f_3)\mathbf{k}$.

Theorem 3.3.12. *curl*$(\boldsymbol{f}+\boldsymbol{g}) =$ *curl*$\boldsymbol{f}+$ *curl*$\boldsymbol{g}$.
That is, $\nabla \times (\boldsymbol{f}+\boldsymbol{g}) = \nabla \times \boldsymbol{g} + \nabla \times \boldsymbol{g}$.

Proof.

$$\begin{aligned} \nabla \times (\mathbf{f}+\mathbf{g}) &= \sum \mathbf{i} \times \frac{\partial}{\partial x}(\mathbf{f}+\mathbf{g}) = \sum \mathbf{i} \times \frac{\partial \mathbf{f}}{\partial x} + \sum \mathbf{i} \times \frac{\partial \mathbf{g}}{\partial x} \\ &= \nabla \times \mathbf{f} + \nabla \times \mathbf{g} \end{aligned}$$

□

Theorem 3.3.13. *curl* $(\boldsymbol{f} \times \boldsymbol{g}) = (\boldsymbol{g}.\nabla)\boldsymbol{f} - (\boldsymbol{f}.\nabla)\boldsymbol{g} + \boldsymbol{f}$ *div* $\boldsymbol{g} - \boldsymbol{g}$ *div* $\boldsymbol{f}$

Proof.

$$\begin{aligned} curl(\mathbf{f}\times\mathbf{g}) = \nabla \times (\mathbf{f}\times\mathbf{g}) &= \sum \mathbf{i} \times \frac{\partial}{\partial x}(\mathbf{f}\times\mathbf{g}) = \sum \mathbf{i} \times \left[\mathbf{f} \times \frac{\partial \mathbf{g}}{\partial x} + \frac{\partial \mathbf{g}}{\partial x} \times \mathbf{g}\right] \\ &= \sum \mathbf{i} \times \left(\mathbf{f} \times \frac{\partial \mathbf{g}}{\partial x}\right) + \sum \mathbf{i} \times \left(\frac{\partial \mathbf{f}}{\partial x} \times \mathbf{g}\right) \\ &= \sum \left[\left(\mathbf{i}.\frac{\partial \mathbf{g}}{\partial x}\right)\mathbf{f} - (\mathbf{i}.\mathbf{f})\frac{\partial \mathbf{g}}{\partial x}\right] + \\ &\qquad \sum \left[(\mathbf{i}.\mathbf{g})\frac{\partial \mathbf{f}}{\partial x} - \left(\mathbf{i}.\frac{\partial \mathbf{f}}{\partial x}\right)\mathbf{g}\right] \\ &= \mathbf{f}\left(\sum \mathbf{i}.\frac{\partial \mathbf{g}}{\partial x}\right) - \mathbf{g}\sum\left(\mathbf{i}.\frac{\partial \mathbf{g}}{\partial x}\right) + \\ &\qquad \sum (\mathbf{i}.\mathbf{g})\frac{\partial \mathbf{f}}{\partial x} - \sum (\mathbf{i}.\mathbf{f})\frac{\partial \mathbf{g}}{\partial x} \\ &= \mathbf{f}\ div\ \mathbf{g} - \mathbf{g}\ div\ \mathbf{f} + (\mathbf{g}.\nabla)\ \mathbf{f} - (\mathbf{f}.\nabla)\ \mathbf{g} \end{aligned}$$

□

Theorem 3.3.14. *div curl* $\boldsymbol{f}$=$\nabla.(\nabla\times\boldsymbol{f})=0$

Proof. Let $\mathbf{f}=f_1\mathbf{i}+f_2\mathbf{j}+f_3\mathbf{k}$

$$\therefore\nabla\times\mathbf{f}=\mathbf{i}\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right)-\mathbf{j}\left(\frac{\partial f_1}{\partial z}-\frac{\partial f_3}{\partial x}\right)-\mathbf{k}\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)$$

$$\begin{aligned}\therefore\nabla.(\nabla\times f)&=\frac{\partial}{\partial x}\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial f_1}{\partial z}-\frac{\partial f_3}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)\\&=\frac{\partial^2 f_3}{\partial x\partial y}-\frac{\partial^2 f_2}{\partial x\partial z}-\frac{\partial^2 f_3}{\partial y\partial x}+\frac{\partial^2 f_1}{\partial y\partial z}+\frac{\partial^2 f_2}{\partial z\partial x}-\frac{\partial^2 f_1}{\partial z\partial y}=0\end{aligned}$$

□

Theorem 3.3.15. *curl grad* $\Phi=\nabla\times(\nabla\Phi)=0$

Proof. $\nabla\Phi=\frac{\partial\Phi}{\partial x}\mathbf{i}+\frac{\partial\Phi}{\partial y}\mathbf{j}+\frac{\partial\Phi}{\partial z}\mathbf{k}$

$$\therefore\nabla\times(\nabla\Phi)=\mathbf{i}\left(\frac{\partial^2\Phi}{\partial y\partial z}-\frac{\partial^2\Phi}{\partial z\partial y}\right)-\mathbf{j}\left(\frac{\partial^2\Phi}{\partial x\partial z}-\frac{\partial^2\Phi}{\partial z\partial x}\right)+\mathbf{k}\left(\frac{\partial^2\Phi}{\partial x\partial y}-\frac{\partial^2\Phi}{\partial y\partial x}\right)=0.$$

□

Theorem 3.3.16. *grad*$(\boldsymbol{f}.\boldsymbol{g})=\boldsymbol{f}\times curl\boldsymbol{g}+\boldsymbol{g}\times curl\boldsymbol{f}+(\boldsymbol{f}.\nabla)\boldsymbol{g}+(\boldsymbol{g}.\nabla)\boldsymbol{f}.$

Proof.

$$\begin{aligned}grad(\mathbf{f.g})+\sum\mathbf{i}\frac{\partial}{\partial x}(\mathbf{f.g})&=\sum\mathbf{i}\left[\mathbf{f}.\frac{\partial\mathbf{g}}{\partial x}+\frac{\partial\mathbf{f}}{\partial x}.\mathbf{g}\right]\\&=\sum\mathbf{f}.\left(\frac{\partial\mathbf{g}}{\partial x}\right)\mathbf{i}+\sum\left(\frac{\partial\mathbf{g}}{\partial x}.\mathbf{g}\right)\mathbf{i}\qquad\cdots(1)\end{aligned}$$

$$\text{Now,}\quad\mathbf{f}\times\left(\mathbf{i}\times\frac{\partial\mathbf{g}}{\partial x}\right)=\left(\mathbf{f}.\frac{\partial\mathbf{g}}{\partial x}\right)\mathbf{i}-(\mathbf{f.i})\frac{\partial\mathbf{g}}{\partial x}$$

$\left(\mathbf{f}.\frac{\partial \mathbf{g}}{\partial x}\right)\mathbf{i} = \mathbf{f} \times \left(\mathbf{i} \times \frac{\partial \mathbf{g}}{\partial x}\right) + (\mathbf{f}.\mathbf{i})\frac{\partial \mathbf{g}}{\partial x}$

$$\begin{aligned}
\sum \mathbf{f}.\left(\frac{\partial \mathbf{g}}{\partial x}\right)\mathbf{i} &= \sum \mathbf{f} \times \left(\mathbf{i} \times \frac{\partial \mathbf{g}}{\partial x}\right) + \sum (\mathbf{f}.\mathbf{i})\frac{\partial \mathbf{g}}{\partial x} \\
&= \mathbf{f} \times \left(\sum \mathbf{i} \times \frac{\partial \mathbf{g}}{\partial x}\right) + \left(\sum \mathbf{f}.\mathbf{i}\frac{\partial \mathbf{g}}{\partial x}\right) \\
&= \mathbf{f} \times \left(\sum \mathbf{i} \times \frac{\partial \mathbf{g}}{\partial x}\right) + \left(\mathbf{f}.\sum \mathbf{i}\frac{\partial}{\partial x}\mathbf{g}\right) \\
&= \mathbf{f} \times curl\mathbf{g} + (\mathbf{f}.\nabla)\mathbf{g} \qquad \cdots (2)
\end{aligned}$$

Similarly, $\sum \left(\frac{\partial \mathbf{g}}{\partial x}.\mathbf{g}\right)\mathbf{i} = \mathbf{g} \times curl\ \mathbf{f} + (\mathbf{g}.\nabla)\ \mathbf{f}$ $\cdots (3)$

Substituting (2) and (3) in (1), we get the result. □

Theorem 3.3.17. $\nabla \times (\Phi \boldsymbol{f}) = \nabla\Phi \times f + \Phi(\nabla \times f)$

Proof.

$$\begin{aligned}
curl(\Phi\mathbf{f}) = \nabla \times (\Phi\mathbf{f}) &= \sum \left[\mathbf{i} \times \frac{\partial}{\partial x}(\Phi\mathbf{f})\right] = \sum \left[\mathbf{i} \times \left(\frac{\partial \Phi}{\partial x}\mathbf{f} + \Phi\frac{\partial \mathbf{f}}{\partial x}\right)\right] \\
&= \left[\sum \frac{\partial \Phi}{\partial x} i\right] \times \mathbf{f} + \Phi \sum \left(\mathbf{i} \times \frac{\partial \mathbf{f}}{\partial x}\right) \\
&= \nabla\Phi \times \mathbf{f} + \Phi(\nabla \times \mathbf{f})
\end{aligned}$$

□

Theorem 3.3.18. *$curl(curl\boldsymbol{f}) = grad\ div\boldsymbol{f} - \nabla^2\boldsymbol{f}$*

That is, $\nabla \times (\nabla \times \boldsymbol{f}) = \nabla(\nabla.\boldsymbol{f}) - \nabla^2\boldsymbol{f}$

Proof.

$$\begin{aligned}
\therefore \nabla\times\mathbf{f} &= \mathbf{i}\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right)-\mathbf{j}\left(\frac{\partial f_1}{\partial z}-\frac{\partial f_3}{\partial x}\right)-\mathbf{k}\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)\\
\nabla\times(\nabla\times\mathbf{f}) &= \sum\left\{\frac{\partial}{\partial y}\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)-\frac{\partial}{\partial z}\left(\frac{\partial f_1}{\partial z}-\frac{\partial f_3}{\partial x}\right)\right\}\mathbf{i}\\
&= \sum\left\{\left(\frac{\partial^2 f_2}{\partial y\partial x}+\frac{\partial^2 f_3}{\partial z\partial x}\right)-\left(\frac{\partial^2 f_1}{\partial y^2}+\frac{\partial^2 f_1}{\partial z^2}\right)\right\}\mathbf{i}\\
&= \sum\left\{\frac{\partial}{\partial x}\left(\frac{\partial f_2}{\partial x}+\frac{\partial f_1}{\partial y}\right)-\left(\frac{\partial^2 f_1}{\partial y^2}+\frac{\partial^2 f_1}{\partial z^2}\right)\right\}\mathbf{i}\\
&= \sum\left\{\frac{\partial}{\partial x}\left(\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}\right)-\left(\frac{\partial f_1^2}{\partial x^2}+\frac{\partial f_1^2}{\partial y^2}+\frac{\partial f_1^2}{\partial z^2}\right)\right\}\mathbf{i}\\
&= \sum\left\{\frac{\partial}{\partial x}(\nabla.\mathbf{f})-(\nabla^2 f_1)\right\}\mathbf{i}\\
&= \sum\left\{\frac{\partial}{\partial x}(\nabla.\mathbf{f})\mathbf{i}\right\}-\sum(\nabla^2 f_1)\mathbf{i}\\
&= \nabla(\nabla.\mathbf{f})-\nabla^2\mathbf{f}
\end{aligned}$$

□

Definition 3.3.19. *A vector* $\boldsymbol{f}$ *is called a* ***harmonic vector*** *if* $\nabla^2\boldsymbol{f}=0$.

Corollary 3.3.20. *If* $\boldsymbol{f}$ *is a harmonic vector, then* $\nabla\times(\nabla\times\boldsymbol{f})=\nabla(\nabla.\boldsymbol{f})$

Proof. $\nabla\times(\nabla\times\mathbf{f})=\nabla(\nabla.\mathbf{f})-\nabla^2\mathbf{f}=\nabla(\nabla.\mathbf{f})$ (since $\mathbf{f}$ is harmonic) □

3.3.1 Solved problems

Problem 3.3.21. *Find curl curl* $\boldsymbol{f}$ *at the point* $(1,1,1)$ *if* $\boldsymbol{f}=x^2y\boldsymbol{i}+xz\boldsymbol{j}+2yz\boldsymbol{k}$

Solution.

$$curl\ \mathbf{f}=\nabla\times\mathbf{f}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ x^2y&xz&2yz\end{vmatrix}=(2z-x)\mathbf{i}+(z-x^2)\mathbf{k}$$

$$\therefore curl\ curl\ \mathbf{f}=\nabla\times(\nabla\times\mathbf{f})=(2x+2)\mathbf{j}.$$

$$\therefore \text{At}\ \ (1,1,1),\quad \nabla\times(\nabla\times\mathbf{f})=4\mathbf{j}$$

Problem 3.3.22. *Prove that div*$\boldsymbol{r} = 3$ *and curl* $\boldsymbol{r} = 0$ *where* $\boldsymbol{r}$ *is the poistion vector of a point* (x, y, z) *in space.*

Solution. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

$$\begin{aligned}
div\,\mathbf{r} &= \nabla.\mathbf{r} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right).(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \\
&= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3 \\
curl\,\mathbf{r} &= \nabla \times \mathbf{r} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \\
&= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = 0
\end{aligned}$$

Problem 3.3.23. *Prove that* $div(r^n\boldsymbol{r}) = (n+3)r^n$. *Deduce that* $r^n\boldsymbol{r}$ *is solenoidal if and only if* $n = -3$.

Solution. $r^n\mathbf{r} = r^n(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$.

$$\begin{aligned}
\therefore div\ (r^n\mathbf{r}) &= \frac{\partial}{\partial x}(xr^n) + \frac{\partial}{\partial y}(yr^n) + \frac{\partial}{\partial z}(zr^n) \\
&= r^n + xnr^{n-1}\frac{\partial r}{\partial x} + r^n + ynr^{n-1}\frac{\partial r}{\partial y} + r^n + znr^{n-1}\frac{\partial r}{\partial z} \\
&= 3r^n + nr^{n-2}(x^2 + y^2 + z^2) \quad \left(\text{since } \frac{\partial r}{\partial x} = \frac{x}{r} \text{ etc}\right) \\
&= 3r^n + nr^{n-2}r^2 \\
&= (3+n)r^n
\end{aligned}$$

Now, $r^n\mathbf{r}$ is solenoidal if and only if *div* $r^n\mathbf{r} = 0$. That is, if and only if $(3+n)r^n = 0$ That is, if and only if $n = -3$.

Problem 3.3.24. *Show that the vector* $\boldsymbol{f} = (y^2 - z^2 + 3yz - 2x)\boldsymbol{i} + (3xz + 2xy)\boldsymbol{j} + (3xy - 2xz + 2z)\boldsymbol{k}$ *is both irrotational and solenoidal.*

Solution. Let $\mathbf{f} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$ where

$f_1 = (y^2 - z^2 + 3yz - 2x); f_2 = (3xz + 2xy); f_3 = (3xy - 2xz + 2z)$

$$\therefore \, div\, \mathbf{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = -2 + 2x - 2x + 2 = 0$$

Also, $curl\, \mathbf{f} = 0$ (verify). Hence $\mathbf{f}$ is irrotational.

Hence $\mathbf{f}$ is both irrotational and solenoidal.

Problem 3.3.25. *If $\boldsymbol{f}$ is solenoidal, prove that curl curl curl curl $\boldsymbol{f} = \nabla^4 \boldsymbol{f}$*

Solution.

$$\begin{aligned}
curl \;\; curl \;\; curl \;\; curl\, \mathbf{f} &= \nabla \times \nabla \times \nabla \times \nabla \times \mathbf{f} \\
&= \nabla \times \nabla \times [\nabla(\nabla.\mathbf{f}) - \nabla^2 \mathbf{f})] \\
&= \nabla \times \nabla \times (-\nabla^2 \mathbf{f}) \;\; (\text{since } \mathbf{f} \text{ is solenoidal } \nabla.\mathbf{f} = 0) \\
&= \nabla \times \nabla \times \mathbf{g} \text{ where } \mathbf{g} = -\nabla^2 \mathbf{f} \\
&= \nabla(\nabla.\mathbf{g}) - \nabla^2 \mathbf{g} \\
&= -\nabla^2 \mathbf{g} \;\; [\text{since } \nabla.\mathbf{g} = \nabla.(-\nabla^2 \mathbf{f}) = \nabla^2(\nabla.\mathbf{f}) = 0] \\
&= -\nabla^2(-\nabla^2 \mathbf{f}) \\
&= \nabla^4 \mathbf{f}.
\end{aligned}$$

Problem 3.3.26. *If $\varphi(x, y, z)$ is any solution of Laplace's equation, prove that $\nabla\varphi$ is both solenoidal and irrotational.*

Solution. Since φ is a solution of Laplace equation, we have $\nabla^2\varphi = 0$ $\cdots (1)$

Now, $div(\nabla\varphi) = \nabla.(\nabla\varphi) = \nabla^2\varphi$ =0 (by(1)). Hence $\nabla\varphi$ is solenoidal.

Now, $curl(\nabla\varphi) = \nabla \times (\nabla\varphi) = 0$ (verify)

$\therefore \nabla\varphi$ is irrotational. Hence the result.

Problem 3.3.27. *Prove that curl$(\boldsymbol{r} \times \boldsymbol{a}) = -2\boldsymbol{a}$ where $\boldsymbol{a}$ is a constant vector.*

Solution.

$$
\begin{aligned}
curl(\mathbf{r}\times\mathbf{a}) &= \nabla\times(\mathbf{r}\times\mathbf{a}) = \sum\left[\mathbf{i}\times\frac{\partial}{\partial x}(\mathbf{r}\times\mathbf{a})\right]\\
&= \sum\left[\mathbf{i}\times\left(\frac{\partial \mathbf{r}}{\partial x}\times\mathbf{a}+\mathbf{r}\times\frac{\partial \mathbf{a}}{\partial x}\right)\right]\\
&= \sum\left[\mathbf{i}\times\left(\frac{\partial \mathbf{r}}{\partial x}\times\mathbf{a}\right)\right] \quad \text{(since } \mathbf{a} \text{ is a constant vector)}\\
&= \sum[\mathbf{i}\times(\mathbf{i}\times\mathbf{a})] \quad \left(\text{since } \frac{\partial \mathbf{r}}{\partial x}=\mathbf{i}\right)\\
&= \sum[(\mathbf{i.a})\mathbf{i}\ \text{-}\ (\mathbf{i.a})\mathbf{a}] = \sum[(\mathbf{i.a})\mathbf{i}\ \text{-}\ \mathbf{a}]\\
&= [(\mathbf{i.a})\mathbf{i}\ \text{-}\ \mathbf{a}]+[(\mathbf{j.a})\mathbf{j}\ \text{-}\ \mathbf{a}]+[(\mathbf{k.a})\mathbf{k}\ \text{-}\ \mathbf{a}]\\
&= (\mathbf{i.a})\mathbf{i}+(\mathbf{j.a})\mathbf{j}+(\mathbf{k.a})\mathbf{k}\ \text{-}\ \mathbf{3a} = \mathbf{a}\ \text{-}\ \mathbf{3a}\\
&= \mathbf{\text{-}2a}.
\end{aligned}
$$

Exercises 3.3.28. 1. If $\nabla\phi = (y + sin\ z)\mathbf{i} + x\mathbf{j} + xcosz\mathbf{k}$, find $\phi(x, y, z)$.

2. Show that $div(\frac{\mathbf{r}}{r}) = \frac{2}{r}$.

3. If $\mathbf{f} = x^2zi - 2y^3z^2j + xy^2zk$ find

   (i) div $\mathbf{f}$ (ii) $curl$ $\mathbf{f}$ at $(1, -1, 1)$

4. Find divergence and curl of the vector

   (i) (xyz^2, yzx^2, zxy^2) (ii) $(x\cos z, ylogx, -z^2)$

5. Show that $\Phi = ax^2 + by^2 + cz^2$ satisfies laplace's equation if $a + b + c = 0$.

6. Prove that $(\mathbf{f}\times\nabla)\times\mathbf{r} = -2\mathbf{f}$ where $\mathbf{r}$=x$\mathbf{i}$+y$\mathbf{j}$+z$\mathbf{k}$.

7. Prove that the vector (i) $(3y^4z^2, 4x^3z^2, -3x^2y^2)$ is solenoidal.

   (ii) $(x^2 - yz, y - zx, z^2 - xy)$ is irrotational.

8. Let $\mathbf{f}$ be a vector valued function and ϕ be a scalar valued function. Prove that $div(\phi\mathbf{f}) = (grad\phi).\mathbf{f} + (div\mathbf{f})\phi$.

9. If $\mathbf{f} = (ax + 3y + 4z)\mathbf{i} + (x - 3y + 3z)\mathbf{j} + (3x + 2y - z)\mathbf{k}$ is solenoidal, find the constant a.

10. Prove that $div\left(\mathbf{a}\times grad\frac{1}{\mathbf{r}}\right) = 0$, where $\mathbf{a}$ is a constant vector.

Chapter 4

UNIT IV

4.1 LINE AND SURFACE INTEGRALS

4.1.1 INTRODUCTION

In this chapter, introduce the concept of line and surface integrals leading to the theorems of Green, Stokes and Gauss which express these integrals as a certain double or triple as the case may be.

4.1.2 LINE INTEGRALS

Another way of generalising the Riemann integral $\int_a^b f(x)dx$ is by replacing the interval $[a,b]$ by a curve in R^3. In this generalisisation the integrand is vector valued function $f = f + 1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$.

Definition 4.1.1. *Let C be a curve in $\boldsymbol{R}^3$ described by a continuous vector valued function $\boldsymbol{r} = \boldsymbol{r}(t) = s(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}$ where $a \leq t \leq b$.*

*Let $\boldsymbol{f} = f_1(x,y,z)\boldsymbol{i} + f_2(x,y,z)\boldsymbol{j} + f_3(x,y,z)\boldsymbol{k}$ be a continuous function defined in some region which contains the curve C. The **line integral** of $\boldsymbol{f}$ over C denoted by $\int_C \boldsymbol{f}.d\boldsymbol{r}$ is defined by*

$$\int_C \boldsymbol{f}.dr = \int_a^b [f_1[x(t),y(t),z(t)]x'(t) + f_2[x(t),y(t),z(t)]y'(t) + f_3[x(t),y(t),z(t)]z'(t)]dt$$

Work done by a force

A force is said to do work when its point of application moves. When a particle acted on by a force **f**, move from a point r to a neighbouring point $r+\Delta r$, the work done in this small displacement is defined to be the scalar product $\mathbf{f}.\Delta\mathbf{r}$. If the particle describes an are C, then the work done is given by the line integral $\int\limits_C \mathbf{f}.d\mathbf{r}$

4.1.3 Solved problems

Problem 4.1.2. *Evaluate* $\int\limits_C \boldsymbol{f}.d\boldsymbol{r}$ *where* $\boldsymbol{f} = (x^2+y^2)\boldsymbol{i} + (x^2-y^2)j$ *and* C *is the curve* $y = x^2$ *joining (0,0) and (1,1).*

Solution. The parametric equation of the curve can be taken as $x = t; y = t^2$ where $0 \le t \le 1$.

$$\begin{aligned}\int \mathbf{f}.d\mathbf{r} &= \int\limits_0^1 [(t^2+t^4)1 + (t^2-t^4)2t]dt \\ &= \left[\frac{1}{3}t^3 + \frac{1}{5}t^5 + \frac{1}{2}t^4 - \frac{1}{3}t^6\right]_0^1 = \frac{9}{10}\end{aligned}$$

Problem 4.1.3. *If* $\boldsymbol{f} = x^2\boldsymbol{i} - xy\boldsymbol{j}$ *and* C *is the straight line joining the points (0,0) and (1,1), find* $\int\limits_C \boldsymbol{f}.d\boldsymbol{r}$.

Solution. The equation of the given line is $y = x$ and its parametric equaion can be takes as $x = t, y = t$ where $0 \le t \le 1$.
$\therefore \int\limits_C \mathbf{f}.d\mathbf{r} = \int\limits_0^1 (t^2 - t^2) = 0$

Problem 4.1.4. *Evaluate* $\int \boldsymbol{f}.d\boldsymbol{r}$ *where* $\boldsymbol{f} = (x^2+y^2)i - 2xyj$ *and the curve* C *is the rectangle in the* $x-y$ *plane bounded by* $y = 0, y = b, x = 0, x = a$.

Solution. Let $\mathbf{O} = (0,0), A = (0,a), B = (a,b)$ and $C = (0,b)$ be the vertices of the given rectangle.

Hence $\int\limits_C \mathbf{f}.d\mathbf{r} = \int\limits_{OA} \mathbf{f}.d\mathbf{r} + \int\limits_{AB} \mathbf{f}.d\mathbf{r} + \int\limits_{BC} \mathbf{f}.d\mathbf{r} + \int\limits_{CO} \mathbf{f}.d\mathbf{r}$
Now the parametric equation of OA can be taken as $x = t, y = 0$ where $0 \le t \le a$.

$\therefore \int\limits_{OA} \mathbf{f}.d\mathbf{r} = \int_0^a t^2 dt = \frac{1}{3}a^3$

$\int\limits_{AB} \mathbf{f}.d\mathbf{r} = \int\limits_0^b (-2at)dt$ (since $x = a, y = t$ and $0 \le t \le b$ along AB)

$= -ab^2$

$\int\limits_{BC} \mathbf{f}.d\mathbf{r} = -\int\limits_{CB} \mathbf{f}.d\mathbf{r}$

$= \int\limits_0^a (t^2 + b^2)dt$ (since $x = t, y = b$ and $0 \le t \le b$ along CB)

$= -(\frac{1}{3}a^3 + ab^2)$

$\int\limits_{CO} \mathbf{f}.d\mathbf{r} = -\int\limits_{OC} \mathbf{f}.d\mathbf{r}$

$= -\int_0^b 0\ dt$

$= 0$ (since $x = 0, y = t$ and $0 \le t \le b$ along OC)

$\therefore \int\limits_C \mathbf{f}.d\mathbf{r} = \frac{1}{3}a^3 - ab^2 - (\frac{1}{3} + ab^2) + 0$

$= -2ab^2.$

Problem 4.1.5. *If $\boldsymbol{f} = (2y+3)\boldsymbol{i} + xz\boldsymbol{j} + (yz - x)\boldsymbol{k}$, evaluate $\int\limits_C \boldsymbol{f}.d\boldsymbol{r}$ along the following paths C.*

(i) $x = 2t^2; y = t; z = t^3$ from $t = 0$ to 1.

(ii) The polygonal path $\boldsymbol{P}$ consisting of the three line segments AB, BC and CD where $A = (0,0,0), B = (0,0,1), C = (0,1,1)$ and $D = (2,1,1)$.

(iii) The straight line joining $(0,0,0)$ and $(2,1,1)$.

Solution.

$$\begin{aligned}
(i) \int\limits_C \mathbf{f}.d\mathbf{r} &= \int\limits_0^1 [(2t+3)4t + 2t^5 + (t^4 - 2t^2)3t^2]dt \\
&= [\frac{8}{3}t^3 + 6t^2 + \frac{1}{3}t^6 + \frac{3}{7}t^7 - \frac{6}{5}t^5]_0^1 \\
&= \frac{8}{3} + 6 + \frac{1}{3} + \frac{3}{7} - \frac{6}{5} = \frac{288}{85}
\end{aligned}$$

$$
\begin{aligned}
(ii) \int_P \mathbf{f}.d\mathbf{r} &= \int_{AB} \mathbf{f}.d\mathbf{r} + \int_{BC} \mathbf{f}.d\mathbf{r} + \int_{CD} \mathbf{f}.d\mathbf{r} \\
\int_{AB} \mathbf{f}.d\mathbf{r} &= \int_0^1 0\ dt = 0 \ \ (\text{since } x = 0; y = 0; z = t \text{ and } 0 \le t \le 1 \text{ along } AB) \\
\int_{BC} \mathbf{f}.d\mathbf{r} &= \int_0^1 0\ dt = 0 \ \ (\text{since } x = 0; y = t; z = 1 \text{ and } 0 \le t \le 1 \text{ along } BC) \\
\int_{CD} \mathbf{f}.d\mathbf{r} &= \int_0^2 5\ dt = 0 \ \ (\text{since } x = 1; y = 1; z = t \text{ and } 0 \le t \le 2 \text{ along } CD) \\
&= \left[5t^2\right]_0^2 = 10.
\end{aligned}
$$

Hence $\int\limits_P \mathbf{f}.d\mathbf{r} = 10$.

(iii) The parametric equation of the line joining $(0, 0, 0)$ and $(2, 1, 1)$ can be takes as $x = 2t, y = t, z = t$ where $0 \le t \le 1$.

$$
\begin{aligned}
\therefore \int \mathbf{f}.d\mathbf{r} &= \int_0^1 [(2t + 3)2 + 2t^2 + (t^2 - 2t)]dt \\
&= \int_0^1 (3t^2 + 2t + 6)dt = [t^3 + t^2 + 6t]_0^1 \\
&= 8
\end{aligned}
$$

Problem 4.1.6. *Find the work done by the force* $\boldsymbol{F} = 3xy\boldsymbol{i} - 5z\boldsymbol{j} + 10x\boldsymbol{k}$ *along the curve* C, $x = t^2, y = 2t^2, z = t^3$ *from t=1 to t=2.*

Solution. $\mathbf{F}.d\mathbf{r} = (3xy\mathbf{i} - 5z\mathbf{j} + 10x\mathbf{k}).(dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k})$

$= 3xydx - 5zdy + 10xdz$

$$\text{Total work done } \int_c \mathbf{F}.d\mathbf{r} = \int_C 3xydx - 5zdy + 10xdz$$

$$= \int_1^2 3(t^2+1)(2t^2)2tdt - 5t^3(4t)dt + 10(t^2+1)(3t^2)dt$$

$$= \int_1^2 [(12t^5 + 12t^3) - 20t^4 + (30t^4 + 30t^2)]dt$$

$$= \int_1^2 (12t^5 + 10t^4 + 12t^3 + 30t^2)dt$$

$$= [2t^6 + 2t^5 + 3t^4 + 10t^3]_1^2$$

$$= 320 - 17 = 303$$

Exercises 4.1.7. 1. Evaluate $\int_{(1,1)}^{(4,2)} \mathbf{f}.d\mathbf{r}$ where $f = (x+y)\mathbf{i} + (y-x)\mathbf{j}$ along

(i) the parabola $y^2 = x$;

(ii) The straight line joining (1,1) and (4,2).

2. Evaluate $\int \mathbf{f}.d\mathbf{r}$ where $f = (2x - y + 4)\mathbf{i} + (5y + 3x - 6)\mathbf{j}$ where C is the boundary of the $\triangle ABC$ in the $x - y$ plane with vertices at $A(0,0), B(3,0)$ and $C(3,2)$ traversed in anticlockwise direction.

3. If $\mathbf{f} = (x^2 - y^2)\mathbf{i} + 2xy\mathbf{j}$ evaluate $\int_C \mathbf{f}.d\mathbf{r}$ along the curve C in the $x - y$ plane given by $y = x^2 - x$ from the point $(1,0)$ to $(2,2)$.

4. If $\mathbf{f} = (3x - 2y)\mathbf{i} + (y + 2z)\mathbf{j} - x^2\mathbf{k}$ evaluate $\int_c \mathbf{f}.d\mathbf{r}$ from $(0,0,0)$ to $(1,1,1)$ where C is a path consisting of

   (i) the curve $x = t, y = t^2, z = t^3$;

   (ii) the straight line joining $(0,0,0)$ to $(1,1,1)$.

5. Find the total work done in moving a particle in a field of force $\mathbf{F} = 2xy\mathbf{i} - 3x\mathbf{j} - 5z\mathbf{k}$ along the curve $x = t, y = t^2 + 1$ and $z = 2t^2$ from $t = 0$ to 1.

4.2 SURFACE INTEGRALS

Definition 4.2.1. *Consider a surface* $\boldsymbol{S}$. *Let* $\boldsymbol{n}$ *denote the unit outward normal to the surface* S. *Let* R *be the projection of the surface* $\boldsymbol{S}$ *on the* $x-y$ *plane. Let* $\boldsymbol{f}$ *be a vector valued function defined in some region containing the surface* $\boldsymbol{S}$. *Then the* ***surface integral*** *of* $\boldsymbol{f}$ *over* $\boldsymbol{S}$ *is defined to be*

$$\iint_S \boldsymbol{f.n}dS = \iint_R \frac{\boldsymbol{f.n}}{|\boldsymbol{n.k}|}dx\ dy$$

Note 4.2.2. We can also define surface integral by considering the projection of the surface on the $y-z$ plane or $z-x$ plane.

4.2.1 Solved problems

Problem 4.2.3. *Evaluate* $\iint_S \boldsymbol{f.n}\ dS$ *where* $\boldsymbol{f} = (x+y^2)\boldsymbol{i} - 2x\boldsymbol{j} + 2yz\boldsymbol{k}$ *and* $\boldsymbol{S}$ *is the surface of the plane* $2x+y+2z=6$ *in the first octant.*

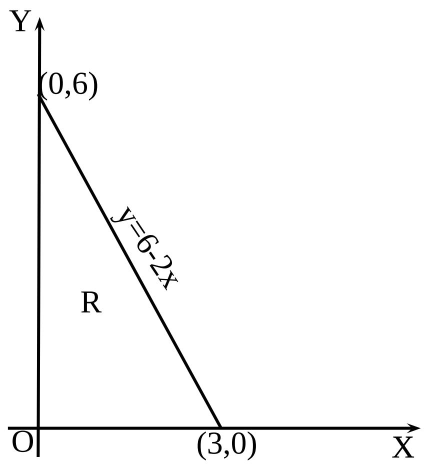
Solution. Let $\varphi(x,y,z) = 2x+y+2z-6$

The unit surface normal $\mathbf{n} = \frac{\nabla\varphi}{|\nabla\varphi} = \frac{\mathbf{2i+j+2k}}{3}$

$$\begin{aligned}\mathbf{f.n} &= \frac{1}{3}[2(x+y^2)-2x+4yz)] \\ &= \frac{1}{3}[2(x+y^2)-2x+2y(6-2x-y)] \\ &= \frac{4}{3}[3y-xy]\end{aligned}$$

Therefore $\frac{\mathbf{f.n}}{|\mathbf{n.k}|} = 2(3y-xy)$

The projection of the surface on the $x-y$ plane is the region R bounded by the axes and straight line $2x+y=6$ as shown in figure.



$$
\begin{aligned}
\therefore \iint_S \mathbf{f.n}\, d\mathbf{S} &= \iint_R 2(3y - xy)dxdy \\
&= 2\int_0^3 \int_0^{6-2x} (3y - xy)dydx \\
&= 2\int_0^3 [\frac{3}{2}y^2 - \frac{1}{2}xy^2]_0^{6-2x} dx \\
&= 2\int_0^3 [\frac{3}{2}(6-2x)^2 - \frac{1}{2}x(6-2x)^2]dx \\
&= [-18(3^2) - 3^4 + 8(3^3) + \frac{1}{2}(6^3)] \\
&= 81
\end{aligned}
$$

Problem 4.2.4. *Evaluate* $\iint_S (\nabla \times \boldsymbol{f}).\boldsymbol{n}\, dS$ *where* $f = y^2\boldsymbol{i} + y\boldsymbol{j} - xz\boldsymbol{k}$ *and* $\boldsymbol{S}$ *is the upper half of the sphere* $x^2 + y^2 + z^2 = a^2$ *and* $z \geq 0$.

Solution. Let $\varphi(x, y, z) = x^2 + y^2 + z^2 - a^2$

The unit surface normal $\mathbf{n}$ is given by

$$\mathbf{n} = \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}}{2\sqrt{x^2+y^2+z^2}}$$

$$=(1/a)(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}).$$

Also $\nabla \times f = z\mathbf{i} - 2y\mathbf{k}$

$\therefore (\nabla \times \mathbf{f}).\mathbf{n} = (1/a)(yz - 2yz) = -(1/a)yz$

Also, $\mathbf{n.k}$=$(1/a)z$

$\therefore \frac{(\nabla\times\mathbf{f})}{|\mathbf{n}.\mathbf{f}|} = -y$

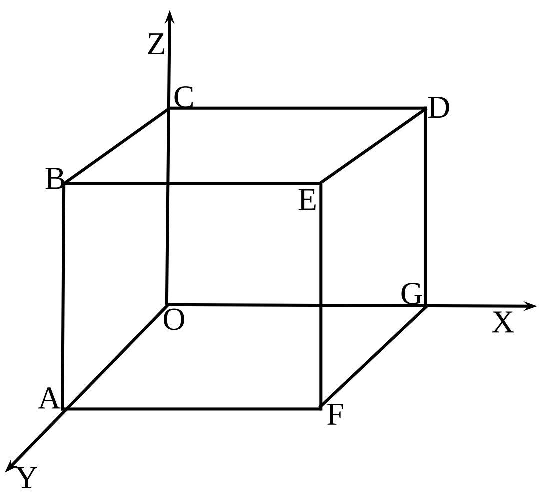
The projection of the surface on the $x-y$ plane is the circle $x^2+y^2=a^2$. Let R denote the interior of the circle.

$\therefore \iint\limits_S (\nabla\times\mathbf{f}).\mathbf{n}d\mathbf{S} = -\iint\limits_R ydxdy$

Put $x = \mathbf{r}\cos\theta$ and $y = \mathbf{r}\sin\theta$. Hence $|J| = \mathbf{r}$

$\therefore \iint(\nabla\times\mathbf{f}).\mathbf{n}d\mathbf{S} = -\int\limits_0^{2\pi}\int\limits_0^{a} r\sin\theta r dr d\theta = -\int\limits_0^{2\pi}\frac{1}{3}a^3\sin\theta d\theta = 0$

Problem 4.2.5. *Evaluate* $\iint \boldsymbol{f}.\boldsymbol{n}\ \ d\boldsymbol{S}$ *where* $\boldsymbol{f} = (x^3-yz)\boldsymbol{i} - 2x^2y\boldsymbol{j} + 2\boldsymbol{k}$ *and* $\boldsymbol{S}$ *is the surface of the cube bounded by* $x=0, y=0, z=0, x=a, y=a$ *and* $z=a$.



Solution. On the face $OABC$, $\mathbf{n} = -\mathbf{i}$ and $x=0$.

$$\begin{aligned}\therefore \iint\limits_{OABC}\mathbf{f}.\mathbf{n}\ d\mathbf{S} &= \int\limits_0^a\int\limits_0^a yzdydz \\ &= \int\limits_0^a \frac{1}{2}a^2zdz \\ &= \frac{1}{4}a^4\end{aligned}$$

On the face $DEFG$, $\mathbf{n} = \mathbf{i}$ and $x = a$

$$\begin{aligned}\therefore \iint_{DEFG} \mathbf{f.n}\ d\mathbf{S} &= \int_0^a \int_0^a (a^3 - yz)\ dy\ dz \\ &= \int_0^a (a^4 - \frac{1}{2}a^2 z)\ dz = (a^5 - \frac{1}{4}a^4)\end{aligned}$$

On the face $OGDC$, $\mathbf{n} = -\mathbf{j}$, $y = 0$.

$$\therefore \iint_{OGDC} \mathbf{f.n}\ d\mathbf{S} = \int_0^a \int_0^a 0dxdz = 0$$

On the face $AFEB$, $\mathbf{n} = \mathbf{j}$ and $y = a$

$$\therefore \iint_{AFEB} \mathbf{f.n}\ d\mathbf{S} = \int_0^a \int_0^a -2x^2 adxdz = \int_0^a -2x^2a^2dx = -\frac{2}{3}a^5$$

On the face $OAFG$, $\mathbf{n} = -\mathbf{k}$, and $z = 0$

$$\therefore \iint_{OAFG} \mathbf{f.n}\ d\mathbf{S} = \int_0^a \int_0^a -2dxdy = -2a^2$$

On the face $CBED$, $\mathbf{n} = \mathbf{k}$, and $z = a$

$$\therefore \iint_{CBED} \mathbf{f.n}\ d\mathbf{S} = \int_0^a \int_0^a -2dxdy = 2a^2$$

$$\therefore \iint_S \mathbf{f.n}\ d\mathbf{S} = \frac{1}{4}a^4 + (a^5 - \frac{1}{4}a^4) + 0 - \frac{2}{3}a^5 - 2a^2 + 2a^2 = \frac{1}{3}a^5$$

Exercises 4.2.6. 1. Evaluate $\iint_S (x^2 + y^2)\ d\mathbf{S}$ where $\mathbf{S}$ is the surface of the cone $z^2 = 3(x^2 + y^2)$ bounded by $z = 0$ and $z = 3$.

2. Evaluate $\iint_S \mathbf{f.n}\ d\mathbf{S}$ where $f = z\mathbf{i} + x\mathbf{j} - 3y^2z\mathbf{k}$ and $\mathbf{S}$ is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$

3. If $f = 4xz\mathbf{i} - y^2\mathbf{j} + yz\mathbf{k}$, evaluate $\iint\limits_S \mathbf{f.n}\ d\mathbf{S}$; $\mathbf{S}$ is the surface of the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0$, and $z = 1$.

4. Evaluate $\iint\limits_S \mathbf{A.n}\ d\mathbf{S}$ where $A = 18a\mathbf{i} - 12\mathbf{j} + 3y\mathbf{k}$ and $\mathbf{S}$ is that part of the plane $2x + 3y + 6z = 12$ which is located in the first octant. That is, $x \geq 0,\ y \geq 0,\ z \geq 0$.

5. Compute $\int\limits_S \mathbf{f}.d\mathbf{S}$, where $f = x^2y\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ over the cylindrical surface, $x^2 + y^2 = 4,\ 0 \leq z \leq 5$ included in the first octant.

6. Compute $\int\limits_S \mathbf{f}.d\mathbf{S}$, where $f = y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$ over the cylindrical surface, $x^2 + y^2 = a^2,\ 0 \leq z \leq h$ included in the first octant.

7. Compute $\int\limits_S \mathbf{f}.d\mathbf{S}$, where $f = yz\mathbf{i} + zx\mathbf{j} + xy\mathbf{k}$ over the entire surface of a sphere $x^2 + y^2 + z^2 = 4$.

8. Compute $\int\limits_S \mathbf{f}.d\mathbf{S}$, where $f = yz\mathbf{i} + zx\mathbf{j} + xy\mathbf{k}$ over the surface of a sphere $x^2 + y^2 + z^2 = 1$ which lies in the first octant.

9. Compute $\int\limits_S \mathbf{f}.d\mathbf{S}$, where $f = y^2z^2\mathbf{i} + z^2x^2\mathbf{j} + x^2y^2\mathbf{k}$ over the surface of a sphere $x^2 + y^2 + z^2 = 1$ above the xy-plane and bounded by this plane.

10. Compute $\int\limits_S \mathbf{f}.d\mathbf{S}$, where $f = yz\mathbf{i} + zx\mathbf{j} + xy\mathbf{k}$ over the entire surface of a sphere $x^2 + y^2 + z^2 = 4$.

11. Compute $\int\limits_S \mathbf{f}.d\mathbf{S}$, where $f = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ over the surface of triangular plane with vertices $(1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$.

4.3 volume integral

A **triple integral** of a function define over a region D in R^3 is denoted by

$\iiint\limits_D f(x, y, z)\ dx\ dy\ dz$ or $\iiint\limits_D f(x, y, z)\ dV$ or $\iiint\limits_D f(x, y, z)\ d(x, y, z)$.

The triple integral can be expressed as an iterated integral in several ways. For example, if a region D in R^3 is given by

D=$\{(x,y,z)|a \le x \le b; \phi_1(x) \le y \le \phi_2(x); \psi_1(x,y) \le z \le \psi_2(x,y)\}$

then

$\iiint\limits_D f(x,y,z)dxdydz = \int\limits_b^a \int\limits_{\phi_1(x)}^{\phi_2(x)} \int\limits_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)dzdydx.$

Note 4.3.1. $\iiint\limits_D dxdydz$ represents the volume of the region D.

Problem 4.3.2. *Evaluate* $I = \int\limits_0^{log\ a} \int\limits_0^{x} \int\limits_0^{x+y} e^{x+y+z}dzdydx.$

Solution.

$$\begin{aligned}
I &= \int\limits_0^{log\ a} \int\limits_0^{x} \left[e^{x+y+z}\right]_0^{x+y} dydx \\
&= \int\limits_0^{log\ a} \int\limits_0^{x} \left[e^{2(x+y)} - e^{x+y}\right] dydx \\
&= \int\limits_0^{log\ a} \left[\frac{e^{2(x+y)}}{2} - e^{x+y}\right]_0^{x} dx
\end{aligned}$$

$$\begin{aligned}
&= \int\limits_0^{log\ a} \left[\frac{e^{4x}}{2} - \frac{3e^{2x}}{2} + e^x\right] dx \\
&= \left[\frac{e^{4x}}{8} - \frac{3e^{2x}}{4} + e^x\right]_0^{log\ a} \\
&= \left[\frac{a^4}{8} - \frac{3a^2}{4} + a - \frac{3}{8}\right]
\end{aligned}$$

Problem 4.3.3. *Evaluate* $\int\limits_0^1 \int\limits_1^3 \int\limits_1^2 xy^2dzdydx$

Solution.

$$\int_0^1\int_1^3\int_1^2 xy^2dzdydx = \int_0^2 xdx\int_1^3 y^3dy\int_1^2 zdz$$
$$= \left[\frac{x^2}{2}\right]_0^2\left[\frac{y^3}{3}\right]_1^3\left[\frac{z^2}{2}\right]_1^2$$
$$= (2-0)\left(9-\frac{1}{3}\right)\left(2-\frac{1}{2}\right)$$
$$= 26$$

Problem 4.3.4. *Express the volume of the sphere $x^2+y^2+z^2=a^2$ as a volume integral and hence evaluate it.*

Solution. Required volume $=2\times$ volume of the hemisphere above the xoy-plane.

$$\text{Required volume} = 2\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} dzdydx$$

$$= 2\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}dydx$$

Taking $a^2 - x^2 = b^2$, when integration with respect to y is performed.

$$\begin{aligned}
V &= 2\int_{-b}^{a}\int_{-b}^{b}\sqrt{b^2-y^2}dydx \\
&= 4\int_{-a}^{a}\int_{0}^{b}\sqrt{b^2-y^2}dydx \quad [\text{since } \sqrt{b^2-y^2} \text{ is an even function of } y.] \\
&= 4\int_{a}^{a}\left(\frac{y}{2}\sqrt{b^2-y^2}+\frac{b^2}{2}sin^{-1}\frac{y}{b}\right)_0^b dx \\
&= \pi\int_{-a}^{a}(a^2-x^2)dx \\
&= 2\pi\left(a^2x-\frac{x^3}{3}\right)_0^a \\
&= \frac{4}{3}\pi a^3
\end{aligned}$$

Problem 4.3.5. *Evaluate* $\iiint(x+y+z)\ dxdydz$ *where* V *is the region of space inside the cylinder* $x^2+y^2=a^2$ *that is bounded by the planes* $z=0$ *and* $z=h$*.*

Solution. The equation $x^2+y^2=a^2$ (in three dimensions (that is in space)) represents the right circular cylinder whose axis is the z-axis and base circle is the one with centre at the origin and radius equal to a.

$$\begin{aligned}
I &= \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{0}^{h}(x+y+z)dzdydx \\
&= \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\left[(x+y)h+\frac{h^2}{2}\right]dydx \\
&= 2h\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}}\left(x+\frac{h}{2}\right)dydx
\end{aligned}$$

[by using porperties of odd and even functions]

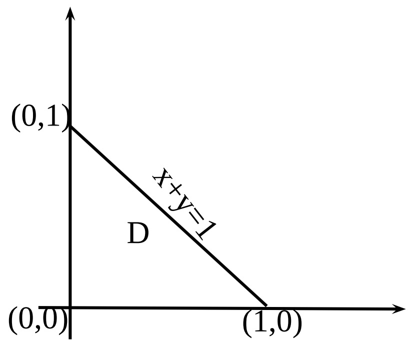
$$\begin{aligned} &= 2h\int_{-a}^{a}\left(x+\frac{h}{2}\right)\sqrt{a^2-x^2}dx \\ &= 2h^2\int_{0}^{a}\sqrt{a^2-x^2}dx \end{aligned}$$

[since $x\sqrt{a^2-x^2}$ is odd and $\sqrt{a^2-x^2}$ is even]

$$\begin{aligned} &= 2h^2\left(\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}sin^{-1}\frac{x}{a}\right)_0^a \\ &= \frac{\pi}{2}a^2h^2 \end{aligned}$$

Problem 4.3.6. *Evaluate* $I = \iiint\limits_D \frac{dxdydz}{(x+y+z+1)^3}$ *where* D *is the region bounded by the planes* $x = 0, y = 0, z = 0$ *and* $x + y + z + 1 = 1$.

Solution. The given region is a tetrahedron. The projection of the given tetrahedron in $x - y$ plane $(z = 0)$ is the triangle bounded by the lines $x = 0, y = 0$ and $x + y = 1$ as shown in the following figure.

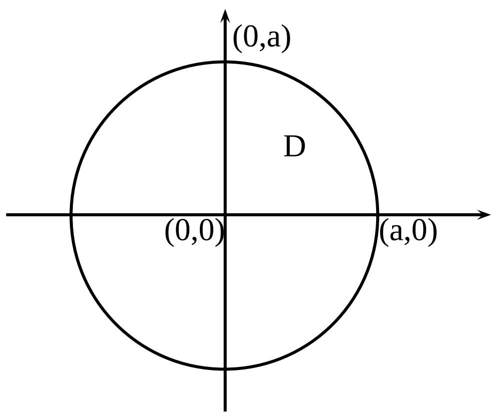


in the given region x varies from 0 to 1. For each fixed x, y varies from 0 to $1 - x$. For each fixed $(x, y), z$ varies from 0 to $1 - x - y$.

$$
\begin{aligned}
\therefore \quad I &= \iiint \frac{dz\ dy\ dx}{(x+y+z+1)^3} \\
&= -\frac{1}{2}\int_0^1 \int_0^{1-x} \left[(x+y+z+1)^{-2}\right]_0^{1-x-x} dy\ dx \\
&= -\frac{1}{2}\int_0^1 \int_0^{1-x} \left[\frac{1}{4} - (x+y+1)^{-2}\right] dy\ dx \\
&= -\frac{1}{2}\int_0^1 \left[\frac{y}{4} + (x+y+1)^{-1}\right]_0^{1-x} dx \\
&= -\frac{1}{2}\int_0^1 \left[\frac{1-x}{4} + \frac{1}{2} - (x+1)^{-1}\right] dx \\
&= -\frac{1}{2}\left[\frac{x}{4} - \frac{x^2}{8} + \frac{x}{2} - log(x+1)\right]_0^1 \\
&= \frac{1}{2}\ log\ 2 - \frac{5}{16}
\end{aligned}
$$

Problem 4.3.7. *Evaluate* $I = \iiint\limits_D xyz\ dx\ dy\ dz$ *where* D *is the region bounded by the poisitve octant of the sphere* $x^2 + y^2 + z^2 = a^2$.

Solution. The projection of the given sphere $x - y$ plane $(z = 0)$ is the region bounded by the circle $x^2 + y^2 = a^2$ and lying in the first quadrant as shown in the following figure.



In the given region x varies form 0 to a. For a fixed x, y varies from 0 to

$\sqrt{a^2-x^2}$. For a fixed (x, y), z varies form 0 to $\sqrt{a^2-x^2-y^2}$

$$\begin{aligned}
\therefore \quad I &= \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xyz\ dz\ dy\ dx \\
&= \frac{1}{2}\int_0^a \int_0^{\sqrt{a^2-x^2}} xy\ (a^2-x^2-y^2)dydx \\
&= \frac{1}{8}\int_0^a x\ (a^2-x^2)\ dx \quad \text{[verify]} \\
&= \frac{1}{16}\left[\frac{1}{3}\ (a^2-x^2)^3\right]_0^a \\
&= \frac{a^6}{48}
\end{aligned}$$

Exercises 4.3.8. 1. Evaluate $\iiint_D (x^2+y^2+z^2)\ dx\ dy\ dz$ where D is the region bounded by the planes $x+y+z=a; x=0; y=0;$ and $z=0$.

2. Evaluate $\iiint_d x^2yz\ dx\ dydz$ where D is the tetrahedron bounded by the planes $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$; $x=0; y=0;$ and $z=0$.

3. Evaluate $\iiint_D xyz\ (x^2+y^2+z^2)\ dx\ dy\ dz$ where D is the positive octant of the sphere $x^2+y^2+z^2=a^2$.

4. Compute $\int_V \mathbf{f}.d\mathbf{V}$ where $\mathbf{f}=2xy\mathbf{i}-x\mathbf{j}+y^2\mathbf{k}$ and V is the region bounded by the surfaces $x=y=z=0$; and $x=y=z=1$.

5. Compute $\int_V \mathbf{f}.d\mathbf{V}$ where $\mathbf{f}=2xy\mathbf{i}-x\mathbf{j}+y^2\mathbf{k}$ and V is the region bounded by the surfaces $x=0, y=0, y=6, z=x^2$, and $z=4$.

6. Compute $\int_V \mathbf{f}.d\mathbf{V}$ where $\mathbf{f}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ and V is the region bounded by $x=0, y=0, z=0$ and $2x+2y+z=4$.

7. Compute $\int_V \mathbf{f}.d\mathbf{V}$ where $\mathbf{f}=4z\mathbf{i}-2x\mathbf{j}+2x\mathbf{k}$ and V is the region bounded by the coordinate planes and the planes $x=y=z=1$.

Chapter 5

UNIT V

5.1 GAUSS, STOKE'S AND GREEN'S THEOREMS

We state without proof the following theorems which connects line and surface integrals with double or triple integrals.

Theorem 5.1.1. *(Green's Theorem in Plane)*
If $\boldsymbol{R}$ *is a closed region of the x-y plane bounded by a simple closed curve* $\boldsymbol{C}$ *and if* M *and* N *are continuous functions of* x *and* y *having continuous partial derivatives in* R *then* $\int_C M\ dx + N\ dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\ dx\ dy$
C *is traversed in the anticlockwise direction.*

Theorem 5.1.2. *(Stoke's Theorem)*
If $\boldsymbol{S}$ *is an open two sided surface bounded by a simple closed curve* $\boldsymbol{C}$ *and* $\boldsymbol{f}$ *is a vector valued function having continuous first order partial derivatives then*
$\int_C \boldsymbol{f}\ .\ d\boldsymbol{r} = \iint_S (\nabla \times \boldsymbol{f})\ \boldsymbol{n}\ dS$ *where* $\boldsymbol{C}$ *is traversed in the anticlockwise direction.*

Theorem 5.1.3. *(Gauss Divergence Theorem)*
If V *is the volume bounded by a closed surfaces* $\boldsymbol{S}$ *and* $\boldsymbol{f}$ *is a vector valued function having continuous partial derivatives then* $\iint_S \boldsymbol{f}.\boldsymbol{n}\ d\boldsymbol{S} = \iiint_V \nabla.\boldsymbol{f}\ dV$

Note 5.1.4. In cartesian form, the Gauss divergence theorem can be written as $\iint_S f_1 \; dy \; dz + f_2 \; dz \; dx + f_3 \; dx \; dy = \iiint_V \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \right) dx \; dy \; dz$

Note 5.1.5. Green's theorem in space is same as Gauss divergence theorem.

5.1.1 Solved problems

Problem 5.1.6. *Verify Green's theorem for the function* $\boldsymbol{f} = (x^2 + y^2)\boldsymbol{i} - 2xy\boldsymbol{j}$ *and* C *is the rectangle in the xy-plane bounded by* $y = 0, y = b, x = 0$ *and* $x = a$.

Solution. Let $\mathbf{f} = (x^2 + y^2)\mathbf{i} - 2xy\mathbf{j} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ where $M(x, y) = x^2 + y^2$ and $N(x, y) = -2xy$.

$\int\limits_C (Mdx + Ndy) = \int\limits_C f.dr = -2ab^2$

Now, $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -(2y + 2y) = -4y$

$$\begin{aligned}
\therefore \quad \iint\limits_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy &= -4 \int\limits_0^b \int\limits_0^a ydxdy = -4 \int\limits_0^b aydy \\
&= -2ab^2. \\
\therefore \quad \iint\limits_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy &= \int\limits_C (Mdx + Ndy)
\end{aligned}$$

Hence Green's theorem is verified.

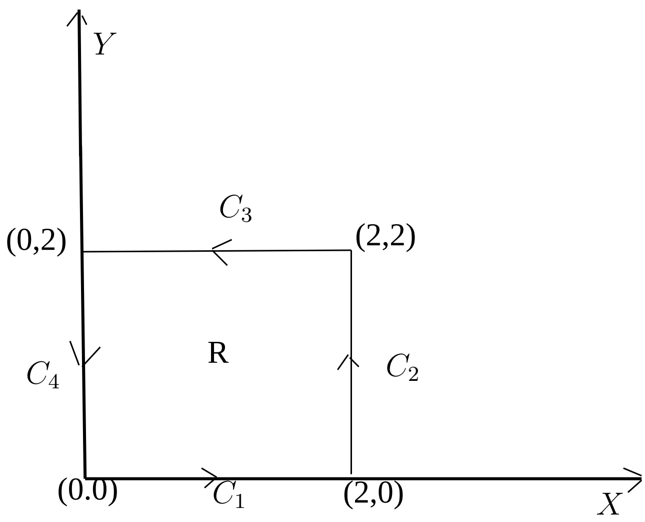
Problem 5.1.7. *Verify Green's theorem for the function* $\boldsymbol{f} = (x - y)\boldsymbol{i} - x^2\boldsymbol{j}$ *and* C *is the boundary of the square* $0 \le x \le 2, 0 \le y \le 2$.

Solution. Let $\mathbf{f} = (x - y)\mathbf{i} - x^2\mathbf{j} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ where $M(x, y) = x - y$ and $N(x, y) = -x^2$.

$\int\limits_C (Mdx + Ndy) = \int\limits_C f.dr$

The boundary C is split into four smooth curves

$C_1(y = 0), \; C_2(x = 2), \; C_3(y = 2), \; C_4(x = 0)$, which are traversed in anticlockwise direction.



$\int\limits_C (Mdx + Ndy) = \int\limits_{C_1} (Mdx + Ndy) + \int\limits_{C_2} (Mdx + Ndy) +$
$\int\limits_{C_3} (Mdx + Ndy) + \int\limits_{C_4} (Mdx + Ndy)$.

On the curve C_1, we have

$\int\limits_{C_1} (Mdx + Ndy) = \int\limits_{C_1} ((x-y)\ dx - x^2\ dy)$

$= \int\limits_{x=0}^{2} x\ dx = 2.$

On the curve C_2, we have

$\int\limits_{C_2} (Mdx + Ndy) = \int\limits_{C_2} ((x-y)\ dx - x^2\ dy)$

$= -4 \int\limits_{y=0}^{2} dy = -8.$

On the curve C_3, we have

$\int\limits_{C_3} (Mdx + Ndy) = \int\limits_{C_3} ((x-y)\ dx - x^2\ dy)$

$= \int\limits_{x=0}^{(} x - 2)\ dx = 2.$

On the curve C_4, we have

$\int\limits_{C_4} (Mdx + Ndy) = \int\limits_{C_4} ((x-y)\ dx - x^2\ dy) = 0$

Hence $\int\limits_C (Mdx + Ndy) = 2 - 8 + 2 + 0 = -4$

Now, $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -2x - (-1) = -2x + 1.$

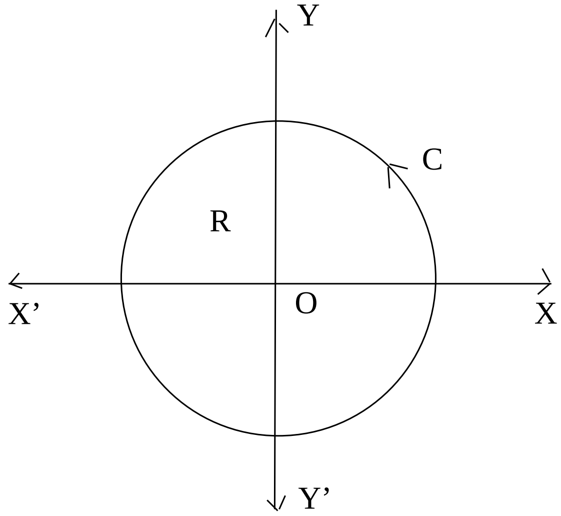
$$\therefore \quad \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \int_0^2 \int_0^2 (1 - 2x)\ dx\ dy = -4$$

$$\therefore \quad \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \int_C (Mdx + Ndy)$$

Hence Green's theorem is verified.

Problem 5.1.8. *Verify Green's theorem for* $\int_C (-y^3 dx + x^3 dy)$ *where* C *is the boundary of the circular region* $x^2 + y^2 = 1$.

Solution. To compute the given integral, we parameterize the circle as follows: $x = cos\ t,\ \ y = sint\ t,\ \ 0 \le t \le 2\pi$



Therefore $\int_C (-y^3 dx + x^3 dy) = \int_0^{2\pi} (sin^4\ t + cos^4\ t) dt$

$$= \int_0^{2\pi} \left(\tfrac{3}{4} + cos\ 4t\right) dt$$

$$= \left[\tfrac{3}{4}t + \tfrac{1}{4} sin\ 4t\right]_0^{2\pi}$$

$$= \frac{3\pi}{2} \qquad \cdots (1)$$

It is given that $M = -y^3$ and $N = x^3$. Therefore

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \iint_R 3(x^2 + y^2) dx\ dy.$$

$$= 3\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x^2+y^2)dy\ dx$$

$$= 6\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (x^2+y^2)dy\ dx$$

$$= 6\int_{-1}^{1}\left[x^2y+\frac{1}{3}y^3\right]_0^{\sqrt{1-x^2}} dx$$

$$= 6\int_{-1}^{1}\left[x^2\sqrt{1-x^2}+\frac{1}{3}(1-x^2)^{\frac{3}{2}}\right] dx$$

$$= 12\int_{0}^{1}\left[x^2\sqrt{1-x^2}+\frac{1}{3}(1-x^2)^{\frac{3}{2}}\right] dx$$

Using the substitution $x = sin\theta$, we get

$$= 12\int_{0}^{\frac{\pi}{2}}\left[sin^2\theta cos^2\theta+\frac{1}{3}cos^4\theta\right] d\theta$$

$$= 12\int_{0}^{\frac{\pi}{2}}\left[(1-cos^2\theta)cos^2\theta+\frac{1}{3}cos^4\theta\right] d\theta$$

$$= 12\int_{0}^{\frac{\pi}{2}}\left[cos^2\theta-\frac{2}{3}cos^4\theta\right] d\theta$$

$$= 12\int_{0}^{\frac{\pi}{2}}\left[\frac{1}{4}+\frac{1}{6}cos\ 2\theta-\frac{1}{12}cos\ 4\theta\right] d\theta$$

$$= \frac{3\pi}{2} \qquad \cdots(2)$$

From (1) and (2), Green's theorem is verified.

Application of Green's theorem to find area.

Let $M = 0$ and $N = x$. Then by Green's theorem, we have

$$\iint_R dx\ dy = \int_C x\ dy. \qquad \cdots(1)$$

The integral on the left is the area of the region R. Let it be denoted by A.

Similarly, we assume that $M = -y$ and $N = 0$. Again by Green's theorem, we find that

$\iint\limits_R \, dx \, dy = -\int\limits_C y \, dx.$ $\cdots(2)$

From (1) and (2), we get

$2A = \iint\limits_R \, dx \, dy = \int\limits_C x \, dy - \int\limits_C y \, dx.$

$A = \frac{1}{2}\left(\int\limits_C x \, dy - \int\limits_C y \, dx\right).$

Problem 5.1.9. *Find the area of the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *using Green's theorem.*

Solution. From the above discussion, the area of the ellipse is given by

$A = \frac{1}{2}\int\limits_C x \, dy - \int\limits_C y \, dx,$

where C is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Its parametric equations are $x = a \, cos \, t, \quad y = bsin \, t, \quad 0 \leq t \leq 2\pi$. Thus

$$A = \frac{1}{2}\left(\int\limits_C x \, dy - \int\limits_C y \, dx\right)$$

$$= \frac{1}{2}\int\limits_0^{2\pi} ab(cos^2t + sin^2t)dt$$

$$= \frac{ab}{2}\int\limits_0^{2\pi} dt$$

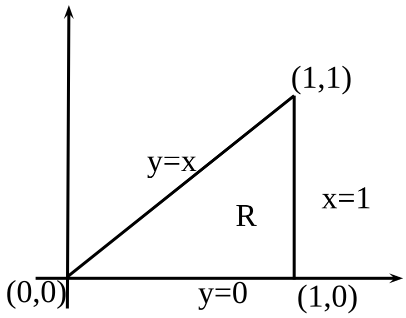
$$= \pi ab$$

Problem 5.1.10. *Using Green's theorem, evaluate* $\int_C (xy - x^2)dx + d^2ydy$ *along the closed curve* C *formed by* $y = 0, x = 1,$ *and* $y = x$

Solution. Green's theorem is $\int\limits_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \, dx \, dy$

Here $M = xy - x^2$ and $N = x^2y$

$\therefore \frac{\partial M}{\partial y} = x$ and $\frac{\partial N}{\partial x} = 2xy$



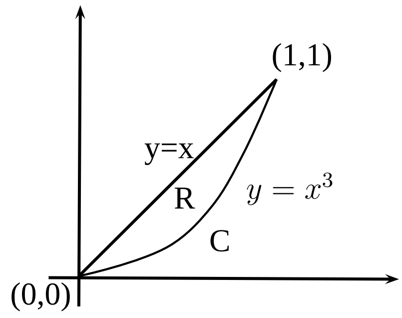
By Green's theorem $\int\limits_C (xy - x^2)dx + x^2ydy = \iint\limits_R (2xy - x)dxdy$ $\cdots(1)$
where R is the region enclose by C (refer the above figure)

$$\begin{aligned}\text{Now,}\quad \iint\limits_R (2xy - x)dxdy &= \int\limits_0^1 \int_y^1 (2xy - x)dxdy \\ &= \int\limits_0^1 \left[x^2y - \frac{x^2}{2}\right]_y^1 dy \\ &= \int\limits_0^1 \left[\left(y - \frac{1}{2}\right) - \left(y^3 - \frac{y^2}{2}\right)\right] dy \\ &= \left[\frac{y^2}{2} - \frac{y}{2} - \frac{y^4}{4} + \frac{y^3}{6}\right]_0^1 \\ &= \frac{1}{2} - \frac{1}{2} - \frac{1}{4} + \frac{1}{6} \\ &= -\frac{1}{12}\end{aligned}$$

Hence from (1), we have $\int_C (xy - x^2)dx + x^2ydy = -\frac{1}{12}$.

Problem 5.1.11. *Using Green's theorem, evaluate* $\int_C (x^2ydx + y^3dy)$ *where* C *is the closed path formed by* $y = x$ *and* $y = x^3$ *from* $(0,0)$ *to* $(1,1)$.

Solution. Green's theorem is $\int\limits_C Mdx + Ndy = \iint\limits_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$
Here $M = x^2y$ and $N = y^3$
$\therefore \frac{\partial N}{\partial x} = 0$ and $\frac{\partial M}{\partial y} = x^2$
By Green's theorem $\int\limits_C (x^2ydx + y^3dy) = \iint\limits_R (-x^2)dxdy$ $\cdots(1)$
where R is the region enclosed by C (refer the figure shown below) .



$$\iint\limits_R (-x^2)dxdy = \int\limits_0^1 \int\limits_0^{y^{1/3}} x^2 dxdy$$
$$= \int\limits_0^1 \left[\frac{x^3}{3}\right]_y^{y1/3} dy$$
$$= -\frac{1}{3}\int\limits_0^1 (y - y^3)dy = -\frac{1}{12} \quad \text{(verify)}$$

Hence from (1), $\int\limits_C (x^2ydx + y^3dy) = -\frac{1}{12}$

Problem 5.1.12. *Verify Stokes theorem for the vector function* $\boldsymbol{f} = y^2\boldsymbol{i} + y\boldsymbol{j} - xz\boldsymbol{k}$ *and* S *is the upper half of the sphere* $x^2 + y^2 + z^2 = a^2$ *and* $z \geq 0$

Solution. We have already proved that $\iint\limits_S (\nabla \times \mathbf{f}).\mathbf{n}\ ds = 0$ Now the boundary C of the hemisphere is given by the equations $x = a\cos\theta, y = a\sin\theta, z = 0, 0 \leq \theta \leq 2\pi$

$$\therefore \int\limits_C \mathbf{f}.d\mathbf{r} = \int\limits_c y^2dx + ydy - xzdz$$
$$= \int\limits_0^{2\pi} [a^2\sin^2\theta(-a\sin\theta) + a\sin\theta(a\cos\theta)]d\theta$$
$$= -a^3\int\limits_0^{2\pi} \sin^3\theta d\theta + a^2\int\limits_0^{2\pi} \sin\theta\cos\theta d\theta$$
$$= 0 \quad \text{(verify)}$$

$$\therefore \int\limits_C \mathbf{f}.d\mathbf{r} = \iint\limits_S (\nabla \times \mathbf{f}).\mathbf{n}dS = 0$$

Hence Stoke's theorem is verified.

Problem 5.1.13. *Verify Stoke's theorem for* $\boldsymbol{f} = (2x - y)\boldsymbol{i} - yz^2\boldsymbol{j} - y^2z\boldsymbol{k}$ *where* S *in the upper half surface of the sphere* $x^2 + y^2 + z^2 = 1$ *and* C *is its boundary.*

Solution. Stokes's theorem is $\int\limits_C \mathbf{f}.d\mathbf{r} = \iint\limits_S (\nabla \times \mathbf{f}).\mathbf{n} \;\; dS$

Here S is the hemisphere $x^2 + y^2 + z^2 = 1$ and C is the circle $x^2 + y^2 = 1, z = 0$

$$\begin{aligned}
\text{We find } \int\limits_C \mathbf{f}.d\mathbf{r} &= \int\limits_C [(2x-y)\mathbf{i} - yz^2\mathbf{j} - y^2z\mathbf{k}].(dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}) \\
&= \int\limits_C (2x-y)dx \quad (\text{since } C \text{ lies on } z = 0) \\
&= \int\limits_0^{2\pi} (2\cos\theta - \sin\theta)(-\sin\theta d\theta) \\
&\qquad\qquad (\text{using parametric equation of the circle } x^2 + y^2 = 1) \\
&= \int\limits_0^{2\pi} [-\sin 2\theta + \sin^2\theta] d\theta \\
&= \left[\frac{\cos 2\theta}{2} + \frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)\right]_0^{2\pi} \quad \left(\text{since } \;\sin^2\theta = \frac{1 - \cos 2\theta}{2}\right) \\
&= \left[\left(\frac{1}{2} + \pi\right) - (12)\right] = \pi
\end{aligned}$$

We evaluate $\iint\limits_S \mathit{curl}\, \mathbf{f}.\mathbf{n}\; dS$

$$\begin{aligned}
\mathit{curl f} &= \left| \begin{array}{ccc} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2z \end{array} \right] \\
&= \mathbf{i}(-2yz + 2yz) - \mathbf{j}(0) + \mathbf{k}(0 + 1) = \mathbf{k}
\end{aligned}$$

The unit surface normal $n = \frac{\nabla\varphi}{|\nabla\varphi|}$ where

$\varphi = x^2 + y^2 + z^2 - 1 = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$\therefore \mathit{curl}\; \mathbf{f}.\mathbf{n} = \mathbf{k}.(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = z$

The projection of S on xy plane is the circular disc R with centre origin and radius 1.

$$\begin{aligned}
\therefore \quad \iint_S curl\mathbf{f}.\mathbf{n}ds &= \iint_R zds = \iint_R z\frac{dxdy}{|\mathbf{n}.\mathbf{k}|} \\
&= \iint_R z\left(\frac{dxdy}{z}\right) \\
&= \iint_R dxdy = \text{Area of the unit circle.} \\
&= \pi
\end{aligned}$$

Hence Stoke's theorem is verified.

Problem 5.1.14. *Verify Stoke's theorem for* $\boldsymbol{f} = (x^2 - y^2)\boldsymbol{i} + 2xy\boldsymbol{j}$ *in the rectangular region* $x = 0, y = 0, x = a, y = b$.

Solution. Stoke's theorem is $\int\limits_C \mathbf{f}.dr = \iint\limits_S (\nabla \times f).\mathbf{n}dS$

Let $O = (0,0), A = (a,0), B = (a,b), C = (0,b)$ be the vertices of the given rectangle.

$$\therefore \int_C \mathbf{f}.d\mathbf{r} = \int_{OA} \mathbf{f}.d\mathbf{r} + \int_{AB} \mathbf{f}.d\mathbf{r} + \int_{BC} \mathbf{f}.d\mathbf{r} + \int_{CO} \mathbf{f}.d\mathbf{r}$$

Therefore the parametric equation of OA can be taken as $x = t, y = 0$ where $0 \leq t \leq a$

$\therefore \int\limits_{OA} \mathbf{f}.d\mathbf{r} = \int_0^a t^2 dt = \frac{a^3}{3}$

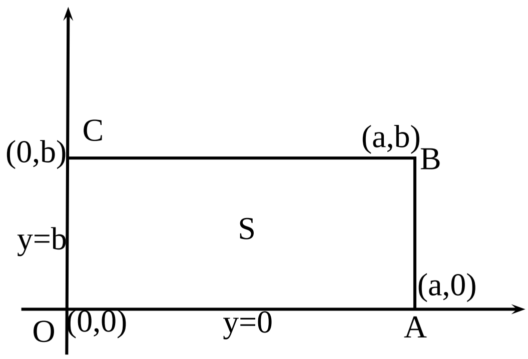
$$\begin{aligned}
\int_{AB} \mathbf{f}.d\mathbf{r} &= \int_0^b 2atdt \quad (\text{since } x = a, y = t \text{ and } 0 \leq t \leq b \text{ along } AB) \\
&= ab^2 \\
\int_{BC} \mathbf{f}.d\mathbf{r} = -\int_{CB} \mathbf{f}.d\mathbf{r} &= -\int_0^a (t^2 - b^2)dt \quad (\text{since } x = t, y = b \text{ and } 0 \leq t \leq a \text{ along } CB) \\
&= -\frac{a^3}{3} + ab^2 \\
\int_{CO} \mathbf{f}.d\boldsymbol{r} &= -\int_{OC} \mathbf{f}.d\mathbf{r} = -\int_0^b 0\ dt \\
&\qquad (\text{since } x = 0, y = t \text{ and } 0 \leq t \leq a \text{ along } OC) \\
&= 0
\end{aligned}$$

$$\text{Thus } \int_C \mathbf{f}.d\mathbf{r} = \frac{a^3}{3} + ab^2 - \frac{a^3}{3} + ab^2 = 2ab^2 \qquad \cdots (1)$$

$$\text{Now, } curl\ \mathbf{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix} = \mathbf{i}(0) - \mathbf{j}(0) + \mathbf{k}(2y + 2y) = 4y\mathbf{k}$$

$$\therefore \iint_S curl\ \mathbf{f.n}\ dS = \iint_R \frac{curl\ \mathbf{f.n}}{|\mathbf{n.k}|}\ dx\ dy$$

Here the surface S denotes the rectangle and unit outward normal $\mathbf{n}$ is $\mathbf{k}$.



$$\begin{aligned} \therefore \iint_S curl\ \mathbf{f.n}\ dS &= \int_0^b \int_0^a 4y\ dx\ dy \\ &= \int_0^b [4xy]_0^a\ dy \\ &= 4a \int_0^b y\ dy \end{aligned}$$

That is, $\iint_S curl\ \mathbf{f}\ .\mathbf{n}\ dS = 2ab^2$ $\cdots (2)$

Thus, from (1) and (2), Stoke's theorem is verified.

Problem 5.1.15. *Evaluate by using Stoke's theorem* $\int_C (yzdx + zxdy + xydz)$ *where* C *is the curve* $x^2 + y^2 = 1, z = y^2$.

Solution. We note that

$yzdx + zxdy + xydz = (yz\mathbf{i} + zx\mathbf{j} + xy\mathbf{k}).(\mathbf{i}dx + \mathbf{j}dy + \mathbf{k}dz)$

$=\mathbf{f}.\mathrm{d}\mathbf{r}$ where $\mathbf{f} = yz\mathbf{i} + zx\mathbf{j} + xy\mathbf{k}$ and $dr = \mathbf{i}dx + \mathbf{j}dy + \mathbf{k}dz$

$\therefore \int\limits_C (yzdx + zxdy + xydz) = \int\limits_C \mathbf{f}.d\mathbf{r} = \iint\limits_S (\nabla \times \mathbf{f}).\mathbf{n}dS$

But $\nabla \times \mathbf{f} = 0$ (verify)

$\therefore \int\limits_C (yzdx + zxdy + xydz) = 0$

Problem 5.1.16. *Evaluate* $\int\limits_C (e^x dx + 2ydy - dz)$ *by using Stoke's theorem where* C *is the curve* $x^2 + y^2 = 4, z = 2.$

Solution. $\int\limits_C (e^x dx + 2ydy - dz) = \int\limits_C \mathbf{f}.d\mathbf{r}$ where $\mathbf{f}$=$e^x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}$ and $d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$.

$= \iint\limits_S (\nabla\times \mathbf{f}).\mathbf{n}\ dS$ (by Stoke's theorem)

where S is any surface whose boundary is given by $x^2 + y^2 = 4$ and $z = 2$.

Now, $\nabla\times\ \mathbf{f}$=0 (verify)

$$\therefore \iint\limits_S (\nabla \times \mathbf{f}).\mathbf{n}\ dS = 0$$

$$\therefore \int\limits_C (e^x dx + 2ydy - dz) = 0$$

Problem 5.1.17. *Using Stoke's theorem, compute* $\int\limits_C \boldsymbol{f}.d\boldsymbol{r}$, *where* $\boldsymbol{f} = (z^2 - y^2 + zx - xy)\boldsymbol{i} + (x^2 - z^2 + xy - yz)\boldsymbol{j} + (y^2 - x^2 + yz - zx)\boldsymbol{k}$ *which is defined in a region of space including a surface* S *whose boundary* C *is the triangle with vertices* $(1, 0, 0),\ (0, 1, 0),\ (0, 0, 1)$.

Solution. By Stoke's theorem, $\int\limits_C \mathbf{f}.d\mathbf{r} = \iint\limits_S (\nabla \times \mathbf{f}).\mathbf{n}dS.$

Given surface is $\phi = x + y + z - 1 = 0$.

$\mathbf{i} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\mathbf{i+j+k}}{\sqrt{3}}$.

curl $\mathbf{f} = 3(y + z)\mathbf{i} + (x + z)\mathbf{j} + (x + y)\mathbf{k}$. Therefore

$$\int\limits_C \mathbf{f}.d\mathbf{r} = \iint\limits_S (\nabla \times \mathbf{f}).\mathbf{n}dS.$$

$$= \iint\limits_{R_{xy}} 6(x + y + z)\ dx\ dy$$

$$= 6 \iint\limits_{R_{xy}} dx\ dy$$

$$= 6 \int\limits_0^1 \int\limits_0^{1-x} dy\ dx$$

$$= 6 \int\limits_0^1 (1-x)\ dx$$

$$= 3$$

Problem 5.1.18. *Verify Gauss divergence theorem for the vector function* $f = (x^3 - yz)\boldsymbol{i} - 2x^2y\boldsymbol{j} + 2\boldsymbol{k}$ *over the cube bounded by* $x = 0, y = 0, a = 0, x = a, y = a$ *and* $z = a$.

Solution. $\therefore \iint\limits_S \mathbf{f.n}\ d\mathbf{S} = \frac{1}{3}a^5$ (refer problem 4.2.5 of section 4.2)
Now $\nabla.\mathbf{f}$=$3x^2 - 2x^2 = x^2$

$$\begin{aligned} \iiint\limits_V \nabla.\mathbf{f}\ dV &= \int\limits_0^a \int\limits_0^a \int\limits_0^a x^2\ dz\ dy\ dx = \frac{1}{3}\int\limits_0^a \int\limits_0^a dydz \\ &= \frac{1}{3}\int\limits_0^a a^4 dz = \frac{1}{3}a^5 \\ \therefore \quad \iint\limits_S \mathbf{f.n}\ d\mathbf{S} &= \iiint\limits_V \nabla.f\ dx\ dy\ dz \end{aligned}$$

Hence Gauss divergence theorem is verified.

Problem 5.1.19. *Verify Gauss divergence theorem for the vector function* $\boldsymbol{f} = y\boldsymbol{i} + x\boldsymbol{j} + z^2\boldsymbol{k}$ *for the cylindrical region* S *given by* $x^2 + y^2 = a^2; z = 0$ *and* $z = h;$

Solution. $\nabla.\mathbf{f} = 2z$

$$\begin{aligned} \therefore \iiint\limits_V \nabla.\mathbf{f}dv &= \int\limits_0^h \int\limits_0^{2\pi} \int\limits_0^a 2zrdrd\theta dz \quad \text{(changing into cylindrical coordinates)} \\ &= \int\limits_0^h \int\limits_0^{2\pi} a^2 z d\theta dz = \int\limits_0^h 2a^2\pi z dz = \pi a^2 h^2 \end{aligned}$$

The surface S of the cylinder consists of a base S_1, the top S_2 and the curved portion S_3.

On S_1, $z = 0$, $\mathbf{n} = -\mathbf{k}$. Hence $\mathbf{f.n} = 0$. Hence $\iint\limits_{S_1} \mathbf{f.n}\ dS = 0$

On S_2, $z = h$, $\mathbf{n} = \mathbf{k}$. Hence $\iint\limits_{S_2}\ f.n\ dS = \iint\limits_{D} h^2\ dx\ dy$ (where D is the region bounded by the circle $x^2 + y^2 = a^2) = \pi h^2 a^2$

$$\text{On}\ \ S_3,\ n = \frac{\nabla\varphi}{|\nabla\varphi|}\ \ \text{where}\ \varphi = x^2 + y^2 - a^2$$

$$= \frac{2xi + 2yj}{2\sqrt{x^2 + y^2}} = \frac{xi + yj}{a}$$

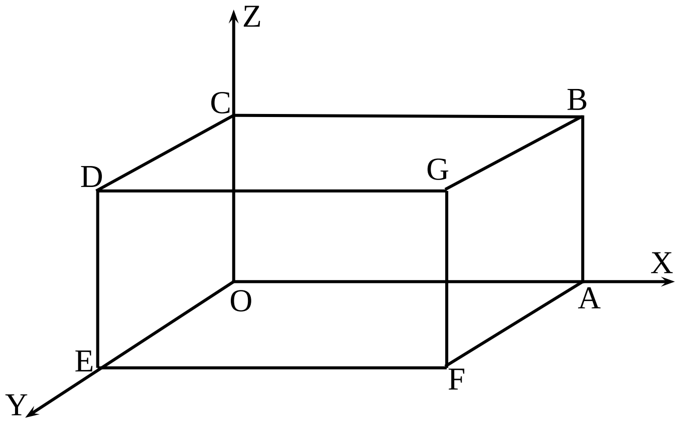
Now $\mathbf{n.j} = \frac{y}{a}$.

$$\begin{aligned}
\therefore\ \frac{\mathbf{f.n}}{|\mathbf{n.j}|} &= 2x \\
\therefore\ \iint\limits_{S_3} \mathbf{f.n} dS &= \iint\limits_{R} 2x\ dy\ dz = a^2 \int\limits_0^b \int_0^{2\pi} 2\cos\theta d\theta dz = 0 \\
\therefore\ \iint\limits_{S} \mathbf{f.n}\ dS &= \iint\limits_{S_1} \mathbf{f.n}\ dS + \iint_{S_2} \mathbf{f.n}\ dS + \iint_{S_3} \mathbf{f.n}\ dS \\
&= \pi h^2 a^2 \\
\therefore\ \iiint\limits_{V} \nabla.\mathbf{f}\ dv &= \iint\limits_{S} \mathbf{f.n}\ dS = \pi h^2 a^2
\end{aligned}$$

Problem 5.1.20. *Verify Gauss divergence theorem for* $f = (x^2 - yz)i + (y^2 - zx)j + (z^2 - xy)k$ *taken over the rectangular parallelopiped,* $0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq c.$

Solution. We first evaluate $\iint_S \mathbf{f.n}$ dS, where S is the surface of the rectangular parallelepiped given by $0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq c$.

It has the following six faces $OABC$ (xz plane); $OAFE$ ($x - y$ plane); $OEDC$(yz plane); $DEFG$(opposite to xz plane); $AFGB$(opposite to yz plane); $BCDG$ (opposite to xy plane).



On the face $OABC$, we have $y = 0$, $\mathbf{n} = -\mathbf{j}$, $0 \le x \le a, 0 \le z \le c$.

$$\begin{aligned}
\therefore \iint\limits_{OABC} \mathbf{f.n}\ \mathrm{d}S &= \int_0^a \int_0^c [(x^2 - 0z)i + (0 - zx)j + (z^2 - 0x)k].(-j)\ \mathrm{d}z\ \mathrm{d}x \\
&= \int_0^a \int_0^c zx\mathrm{d}z\mathrm{d}x \\
&= \int_0^a x \left[\frac{z^2}{2}\right]_0^c \mathrm{d}x \\
&= \frac{c^2}{2}\left[\frac{z^2}{2}\right]_0^a \\
&= \frac{a^2c^2}{4}
\end{aligned}$$

On the face $DEFG$ we have $y = b$, $\mathbf{n} = \mathbf{j}$, $0 \le x \le a, 0 \le z \le c$.

$$\begin{aligned}
\therefore \iint\limits_{DEFG} \mathbf{f.n}\mathrm{d}S &= \int_0^a \int_0^c (b^2 - zx)\mathrm{d}z\mathrm{d}x = \int_0^a \left[b^2 z - \frac{z^2 x}{2}\right]_0^c dx \\
&= \int_0^a \left(b^2 c - \frac{c^2}{2}x\right)\mathrm{d}x = \left[b^2 cx - \frac{c^2x^2}{4}\right]_0^a \\
&= ab^2c - \frac{1}{4}c^2a^2
\end{aligned}$$

On the face $OAFE$, we have $z = 0$, $\mathbf{n} = -\mathbf{k}$, $0 \le x \le a, 0 \le y \le b$.

$$\therefore \iint\limits_{OAFE} \mathbf{f.n}\ dS = \int_0^a \int_0^b xy\ dy\ dx = \int_0^a \left[x\frac{b^2}{2}\right]\ dx = \frac{a^2b^2}{4}$$

On the face $BCDG$, we have $z = c$, $\mathbf{n} = \mathbf{k}$, $0 \leq x \leq a, 0 \leq y \leq b$

$$\begin{aligned}
\therefore \iint\limits_{BCDG} \mathbf{f.n}\ dS &= \int\limits_0^a \int\limits_0^b (c^2 - xy)dy\ dx \\
&= \int\limits_0^a \left[c^2 b - x\frac{b^2}{2}\right]\ dx = ac^2 b - \frac{a^2 b^2}{4}
\end{aligned}$$

Similarly, we can prove that

$$\iint\limits_{OEDC} \mathbf{f.n}\ dS \frac{b^2c^2}{4} \text{ and } \iint\limits_{ABGF} \mathbf{f.n} dS = a^2 bc - \frac{b^2 c^2}{4}$$

$$\therefore \iint\limits_S \mathbf{f.n}\ dS = \frac{a^2c^2}{4} + \left(ab^2c - \frac{1}{4}c^2a^2\right) + \frac{a^2b^2}{4} + \left(ac^2b - \frac{a^2b^2}{4}\right) + \frac{b^2c^2}{4} + \left(a^2bc - \frac{b^2c^2}{4}\right)$$

$$\begin{aligned}
&= ab^2c + ac^2b + a^2bc \\
&= abc(a + b + c) \qquad \cdots (1)
\end{aligned}$$

Now $\nabla.\mathbf{f} = 2x + 2y + 2z$

$$\begin{aligned}
\therefore \iiint\limits_V (\nabla.\mathbf{f})\ dV &= \int\limits_0^a \int\limits_0^b \int\limits_0^c 2(x + y + z)\ dz\ dy\ dx = 2\int\limits_0^a \int\limits_0^b \left(xc + yc + \frac{c^2}{2}\right)\ dy\ dx \\
&= 2\int_0^a \left(xbc + \frac{b^2c}{2} + \frac{c^2b}{2}\right) dx = 2\left[\frac{a^2bc + b^2ca + c^2ba}{2}\right] \\
&= abc(a + b + C) \qquad \cdots (2)
\end{aligned}$$

Therefore from (1) and (2), we get $\iint\limits_S \mathbf{f.n}\ dS = \iiint\limits_V (\nabla.\mathbf{f})\ dV$
Hence Guass divergence theorem is verified.

Problem 5.1.21. *Evaluate* $\iint\limits_S xydydz + y^2dzdx + yzdxdy$ *where* S *is the surface* $x^2 + y^2 + z^2 = a^2$.

Solution. Comparing with the cartesian form of Gauss divergence theorem, we have $f_1 = xy; f_2 = y^2; f_3 = yz$ so that $\mathbf{f} = xy\mathbf{i} + y^2\mathbf{j} + yz\mathbf{k}$.
$\nabla.\mathbf{f} = y + 2y + y = 4y$.

By Gauss divergence theorem

$\iint\limits_S (xy\ dy\ dz + y^2\ dz\ dx + yz\ dx\ dy) = \iiint\limits_V 4ydxdydz$

where V is the volume enclosed by the surface of the sphere

$$\begin{aligned} &= 4\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{-\sqrt{a^2-x^2-z^2}}^{\sqrt{a^2-x^2-z^2}} y\ dy\ dz\ dx \\ &= 4\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 0\ dz\ dx \quad \text{(since y is an odd function)} \\ &= 0 \end{aligned}$$

Problem 5.1.22. *Prove that for a closed surface S, $\iint\limits_S r.n\ dS = 3V$, where V is the volume enclosed by S.*

Solution. By Gauss's divergence theorem $\iint \mathbf{r}.\mathbf{n}dS = \iiint\limits_V \nabla.\mathbf{r}dV$

$$\begin{aligned} &= 3\iiint\limits_V dV \quad \text{(since } \nabla.\mathbf{r} = 3) \\ &= 3V \quad \text{where } V \text{ is the volume enclosed by } S. \end{aligned}$$

Problem 5.1.23. *Show that $\iint \boldsymbol{f}.\boldsymbol{n}\ dS = \iiint\limits_V a^2 dV$ where $r = \varphi a$ and $a = \nabla\varphi$ and $\nabla^2\varphi = 0$.*

Solution. By Gauss divergence theorem, we have

$$\iint_S \mathbf{f}.\mathbf{n}dS = \iiint\limits_V \nabla.\mathbf{f}dV \qquad \cdots(1)$$

$$\begin{aligned} \text{Now } \nabla.\mathbf{f} &= \nabla.(\varphi a) \\ &= \varphi(\nabla.a) + (\nabla\varphi).a \\ &= \varphi(\nabla.a) + a.a = \varphi(\nabla.\nabla\varphi) + a^2 \\ &= \varphi(\nabla^2\varphi) + a^2 = a^2 (\text{since } \nabla^2\varphi = 0) \end{aligned}$$

Therefore from (1), we get $\iint\limits_S \mathbf{f}.\mathbf{n}\ dS = \iiint\limits_V a^2\ dV$.

Exercises 5.1.24. 1. Verify Green's theorem in the plane for $\int\limits_C (x^2 - y^2)\ dx + (y^2 - 2xy)\ dy$ where C is the square with vertices $(0,0), (2,0), (2,2)$ and $(0,2)$.

2. Verify Green's theorem in the plane for $\int\limits_C (xy + y^2)\ dx + x^2\ dy$ where C is the closed curve of the region bounder by $y = x$ and $y = x^2$.

3. verify Stoke's theorem for $\mathbf{f} = 2yi + 3xj - z^2k$ where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 9$.

4. If $\mathbf{f} = xyi + yzj + 3xk$ verify Stoke's theorem for the region bounded by the planes $x = 0, y = 0, z = 0$ and $x + y + z = 1$

5. Verify Guass divergence theorem for the function $\mathbf{f} = 2xzi + yzj + z^2k$ over the upper half of the sphere $x^2 + y^2 + z^2 = a^2$

6. If S is a closed surface enclosing a volume V and if $\mathbf{f}$=$xi + 2yj + 3zk$, prove that $\iint_S \mathbf{f}.\mathbf{n}\ dS = 6V$

7. Evaluate $\int\limits_C \mathbf{f}.d\mathbf{r}$ where $\mathbf{f} = (2y + 3)\mathbf{r} + xz\mathbf{j} + (yz.x)\mathbf{k}$ and the curve C is the straight line joining $(0,0,0)$ and $(2,1,1)$.

8. Evaluate by using Stoke's theorem $\int\limits_C (e^x dx + 2ydy - dz)$ where C is the curve $x^2 + y^2 = 4,\ z = 2$.

9. Evaluate $\iint \mathbf{f}.\mathbf{n}dS$ where $\mathbf{f} = (x^3 - yz)\mathbf{i} - 2x2y\mathbf{j} + 2\mathbf{k}$ and S is the surface of the cube bounded by $x = 0, y = 0, z = 0, x = a, y = a$ and $z = a$.

10. Verify Gauss Divergence theorem for $\mathbf{f} = y\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$ for the cylindrical region S given by $x^2 + y^2 = a^2; z = 0$ and $z = h$.